MATHEMATICS 201-BNK-05
Advanced Calculus
Martin Huard
Winter 2011

Assignment #1
SOLUTIONS

This assignment is due ***Tuesday February 8, 2011*** at the <u>beginning</u> of the class.
Complete solutions with exact answers are expected, presented in a neat and legible manner.

For questions involving Maple, a print-out of your work is expected, where your name is written in the Worksheet, each question is clearly labeled, and the answers are clearly presented. Also, you must copy your file in my "TEST" subfolder (W:\Tests\mhuard\Advanced Calculus \Assignment 1), where your name should be included in the name of the file (for example: Assignment 1 – Your Name).

Question 1 (5 points)

Prove that $\lim_{x\to 2^-} \frac{1}{\sqrt{x^2+5}-3} = -\infty$.

Let $N<0$ be given.

$$\frac{1}{\sqrt{x^2+5}-3} = \frac{1}{\sqrt{x^2+5}-3}\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}$$
$$= \frac{\sqrt{x^2+5}+3}{x^2+5-9}$$
$$= \frac{\sqrt{x^2+5}+3}{(x+2)(x-2)}$$
$$< 6\left(\tfrac{1}{3}\right)\left(\tfrac{-1}{\delta}\right) = \tfrac{-2}{\delta} = N$$

Hyp: $-\delta < x-2 < 0$

$$\frac{1}{x-2} < \frac{-1}{\delta}$$

If $\delta \le 1$, then

$$-1 < x-2 < 0$$
$$1 < x < 2$$
$$1 < x^2 < 4$$
$$6 < x^2+5 < 9$$
$$\sqrt{6}+3 < \sqrt{x^2+5}+3 < 6$$
$$3 < x+2 < 4$$
$$\tfrac{1}{4} < \tfrac{1}{x+2} < \tfrac{1}{3}$$

If $\delta = \min\left\{\frac{-2}{N}, 1\right\}$, then $-\delta < x-2 < 0 \Rightarrow f(x) < N$.

Ergo, $\lim_{x\to 2^-} \frac{1}{\sqrt{x^2+5}-3} = -\infty$

Question 2 (5 points)

Prove that $\lim_{x\to-\infty}\left(1-x^3\right)=4$ is false.

Let $\varepsilon=1$. For every $M<0$, there exist values of x on the interval $(-\infty,M)$ for which $x<-2$.

For these values,

$$\begin{aligned}\left|f(x)-L\right| &= \left|1-x^3-4\right| \\ &= \left|-x^3-3\right| > 3 > \varepsilon = 1\end{aligned}$$

$$\begin{aligned} x &< -2 \\ x^3 &< -8 \\ -x^3 &> 8 \\ -x^3-3 &> 5 \end{aligned}$$

Hence $\lim_{x\to-\infty}\left(1-x^3\right)\neq 4$

Question 3 (5 points)

Prove that the function $f(x)=\dfrac{x}{x^2+4}$ is continuous at $x=-\frac{1}{2}$.

Let $\varepsilon>0$ be given.

$$\begin{aligned}\left|f(x)-f\left(-\tfrac{1}{2}\right)\right| &= \left|\frac{x}{x^2+4}-\frac{\frac{-1}{2}}{\frac{1}{4}+4}\right| \\ &= \left|\frac{x}{x^2+4}+\frac{2}{17}\right| \\ &= \left|\frac{2x^2+17x+8}{17\left(x^2+4\right)}\right| \\ &= \left|\frac{(2x+1)(x+8)}{17\left(x^2+4\right)}\right| \\ &= \left|\frac{2\left(x+\frac{1}{2}\right)(x+8)}{17\left(x^2+4\right)}\right| \\ &< \tfrac{2}{17}8\tfrac{1}{4}\delta = \tfrac{4}{17}\delta = \varepsilon\end{aligned}$$

Hyp: $\left|x+\frac{1}{2}\right|<\delta$

If $\delta\le\frac{1}{2}$, then

$$\begin{aligned} -\tfrac{1}{2} < x+\tfrac{1}{2} &< 1 \\ -1 < x &< 0 \\ 0 < x^2 &< 1 \\ 4 < x^2+4 &< 5 \\ \tfrac{1}{5} < \tfrac{1}{x^2+4} &< \tfrac{1}{4} \\ 7 < x+8 &< 8 \end{aligned}$$

If $\delta=\min\left\{\frac{17}{4}\varepsilon,\frac{1}{2}\right\}$, then $\left|x+\frac{1}{2}\right|<\delta \Rightarrow \left|f(x)-f\left(\frac{-1}{2}\right)\right|<\varepsilon$.

Ergo f is continuous at $x=\frac{-1}{2}$.

Question 4 (5 points)

Prove that the function $f(x)=\begin{cases} x^2+1 & x<1 \\ 3-\frac{1}{2}x & x\geq 1 \end{cases}$ is not continuous at $x=1$.

Let $\varepsilon=\frac{1}{2}$.

Then for all values of δ, there will be values of x on the interval $(1-\delta,1+\delta)$ such that $0<x<1$

For these values,

$$\begin{aligned} |f(x)-f(1)| &= \left|x^2+1-\tfrac{5}{2}\right| \\ &= \left|x^2-\tfrac{3}{2}\right| \\ &> \tfrac{1}{2}=\varepsilon \end{aligned}$$

$$\begin{aligned} 0<x&<1 \\ 0<x^2&<1 \\ -\tfrac{3}{2}<x^2-\tfrac{3}{2}&<\tfrac{-1}{2} \\ \tfrac{1}{2}<-\left(x^2-\tfrac{3}{2}\right)&<\tfrac{3}{2} \\ \tfrac{1}{2}<\left|x^2-\tfrac{3}{2}\right|&<\tfrac{3}{2} \end{aligned}$$

Hence f is not continuous at $x=1$.

Question 5 (5 points)

A sequence $\{a_n\}_{n=0}^{\infty}$ converges to L if $\lim_{n\to\infty} a_n = L$. Give a formal definition for the convergence of a sequence. Note that n is always a nonnegative integer.

Use your definition to prove that the sequence $\left\{\frac{2n}{n+1}\right\}_{n=0}^{\infty}$ converges. Note: you must find the value to which it converges first.

$\{a_n\}_{n=0}^{\infty}\to L$ if and only if $\forall\varepsilon>0,\ \exists N\in\mathbb{Z},\ N>0$ such that $n\in\mathbb{Z},\ n>N \Rightarrow |a_n-L|<\varepsilon$

We have that $\lim_{n\to\infty}\frac{2n}{n+1}=2$

So let us prove that $\left\{\frac{2n}{n+1}\right\}_{n=0}^{\infty}\to 2$.

Let $\varepsilon>0$ be given.

$$\begin{aligned} |a_n-L| &= \left|\tfrac{2n}{n+1}-2\right| \\ &= \left|\tfrac{-2}{n+1}\right| < \tfrac{2}{N+1}=\varepsilon \end{aligned}$$

Hyp: $n>N$

$$\begin{aligned} n+1&>N+1 \\ \tfrac{1}{n+1}&<\tfrac{1}{N+1} \end{aligned}$$

If $N=[\![\tfrac{2}{\varepsilon}-1]\!]+1$. Then $n>N\Rightarrow |a_n-2|<\varepsilon$, ergo $\left\{\frac{2n}{n+1}\right\}_{n=0}^{\infty}\to 2$

Question 5 (25 points)

Consider the vector function $\vec{r}(t)=\left(\int_0^t \cos\left(\frac{\pi}{2}\theta^2\right)d\theta, \int_0^t \sin\left(\frac{\pi}{2}\theta^2\right)d\theta, t\right)$.

a) Reparametrize the curve with respect to arc length measured from the point where $t=0$ in the direction of increasing t.

$$\vec{r}'(t)=\left(\cos\left(\tfrac{\pi}{2}t^2\right),\sin\left(\tfrac{\pi}{2}t^2\right),1\right)$$

$$\left\|\vec{r}'(t)\right\|=\sqrt{\cos^2\left(\tfrac{\pi}{2}t^2\right)+\sin^2\left(\tfrac{\pi}{2}t^2\right)+1}=\sqrt{2}$$

$$s=\int_0^t \left\|\vec{r}'(u)\right\|du=\int_0^t \sqrt{2}du=\sqrt{2}t$$

$$t=\tfrac{\sqrt{2}}{2}s$$

Hence $\vec{r}(s)=\left(\int_0^{\frac{\sqrt{2}}{2}s}\cos\left(\tfrac{\pi}{2}\theta^2\right)d\theta, \int_0^{\frac{\sqrt{2}}{2}s}\sin\left(\tfrac{\pi}{2}\theta^2\right)d\theta, \tfrac{\sqrt{2}}{2}s\right)$

b) Find the curvature κ.

$$\vec{r}''(t)=\left(-\pi t\sin\left(\tfrac{\pi}{2}t^2\right),\pi t\cos\left(\tfrac{\pi}{2}t^2\right),0\right)$$

$$\vec{r}'(t)\times\vec{r}''(t)=\begin{vmatrix} i & j & k \\ \cos\left(\frac{\pi}{2}t^2\right) & \sin\left(\frac{\pi}{2}t^2\right) & 1 \\ -\pi t\sin\left(\frac{\pi}{2}t^2\right) & \pi t\cos\left(\frac{\pi}{2}t^2\right) & 0 \end{vmatrix}$$

$$=\left(-\pi t\cos\left(\tfrac{\pi}{2}t^2\right),-\pi t\sin\left(\tfrac{\pi}{2}t^2\right),\pi t\cos^2\left(\tfrac{\pi}{2}t^2\right)+\pi t\sin^2\left(\tfrac{\pi}{2}t^2\right)\right)$$

$$=\left(-\pi t\cos\left(\tfrac{\pi}{2}t^2\right),-\pi t\sin\left(\tfrac{\pi}{2}t^2\right),\pi t\right)$$

$$\kappa=\frac{\left\|\vec{r}'(t)\times\vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|^3}=\frac{\sqrt{\pi^2t^2\sin^2\left(\frac{\pi}{2}t^2\right)+\pi^2t^2\cos^2\left(\frac{\pi}{2}t^2\right)+\pi^2t^2}}{\left(\cos^2\left(\frac{\pi}{2}t^2\right)+\sin^2\left(\frac{\pi}{2}t^2\right)+1\right)^{\frac{3}{2}}}$$

$$=\frac{\sqrt{2\pi^2t^2}}{2^{\frac{3}{2}}}=\frac{\pi}{2}|t|$$

c) Find the torsion τ (see question 14 in exercise sheet IV).

$$\vec{r}'''(t)=\left(-\pi\sin\left(\tfrac{\pi}{2}t^2\right)-\pi^2t^2\cos\left(\tfrac{\pi}{2}t^2\right),\pi\cos\left(\tfrac{\pi}{2}t^2\right)-\pi^2t^2\sin\left(\tfrac{\pi}{2}t^2\right),0\right)$$

$$\tau(t)=\frac{\left[\vec{r}'(t)\times r''(t)\right]\bullet r'''(t)}{\left\|\vec{r}'(t)\times r''(t)\right\|^2}$$

$$=\frac{\pi^2t\sin\left(\frac{\pi}{2}t^2\right)\cos\left(\frac{\pi}{2}t^2\right)+\pi^3t^3\cos^2\left(\frac{\pi}{2}t^2\right)-\pi^2t\cos\left(\frac{\pi}{2}t^2\right)\sin\left(\frac{\pi}{2}t^2\right)+\pi^3t^3\sin^2\left(\frac{\pi}{2}t^2\right)}{\pi^2t^2\sin^2\left(\frac{\pi}{2}t^2\right)+\pi^2t^2\cos\left(\frac{\pi}{2}t^2\right)+\pi^2t^2}$$

$$=\frac{\pi^3t^3}{2\pi^2t^2}=\frac{\pi}{2}t$$

d) Find the unit tangent vector, unit normal vector and unit binormal vector for the curve at $t=1$.

$$\vec{T}(t)=\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}=\tfrac{1}{\sqrt{2}}\left(\cos\left(\tfrac{\pi}{2}t^2\right),\sin\left(\tfrac{\pi}{2}t^2\right),1\right) \qquad \vec{T}(1)=\left(0,\tfrac{\sqrt{2}}{2},\tfrac{\sqrt{2}}{2}\right)$$

$$\vec{N}(t)=\frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}=\frac{\tfrac{1}{\sqrt{2}}\left(-\pi t\sin\left(\tfrac{\pi}{2}t^2\right),\pi t\cos\left(\tfrac{\pi}{2}t^2\right),0\right)}{\left\|\tfrac{1}{\sqrt{2}}\left(-\pi t\sin\left(\tfrac{\pi}{2}t^2\right),\pi t\cos\left(\tfrac{\pi}{2}t^2\right),0\right)\right\|}$$

$$=\frac{\tfrac{1}{\sqrt{2}}\left(-\pi t\sin\left(\tfrac{\pi}{2}t^2\right),\pi t\cos\left(\tfrac{\pi}{2}t^2\right),0\right)}{\tfrac{1}{\sqrt{2}}\pi t}$$

$$=\left(-\sin\left(\tfrac{\pi}{2}t^2\right),\cos\left(\tfrac{\pi}{2}t^2\right),0\right)$$

$$\vec{N}(1)=(-1,0,0)$$

$$\vec{B}(1)=\vec{T}(1)\times\vec{N}(1)=\begin{vmatrix} i & j & k \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{vmatrix}=\left(0,-\tfrac{\sqrt{2}}{2},\tfrac{\sqrt{2}}{2}\right)$$

e) With Maple, find the curvature and torsion for the curve. Compare with your answer in (b) and (c).
Note: For Maple to simplify the expressions adequately, it has to assume that *t* is a real number. Hence, at the beginning, use the command **assume(t:: real);** Maple will use *t*~ instead of *t* after, to indicate that there is an assumption on *t*.

f) Verify your calculations in (d) with Maple.

g) Sketch a graph of the curve along with the three vectors found in (d) on the same screen with Maple. Note: Use Maple to give you a numerical approximation for $\vec{r}(1)$.

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Assignment 1
Solutions

Question 6

```
restart :
with(VectorCalculus) :
BasisFormat(false) :
assume(t :: real) :
with(plots) :
```

$$r := \left\langle int\left(\cos\left(\frac{\pi}{2}\cdot\theta^2\right), \theta = 0..t\right), int\left(\sin\left(\frac{\pi}{2}\cdot\theta^2\right), \theta = 0..t\right), t\right\rangle$$

$$\begin{bmatrix} \mathrm{FresnelC}(t\sim) \\ \mathrm{FresnelS}(t\sim) \\ t\sim \end{bmatrix} \tag{1}$$

Derivatives

```
r1 := diff(r, t);
r2 := diff(r1, t);
r3 := diff(r2, t);
```

$$\begin{bmatrix} \cos\left(\frac{1}{2}\pi t\sim^2\right) \\ \sin\left(\frac{1}{2}\pi t\sim^2\right) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -\sin\left(\frac{1}{2}\pi t\sim^2\right)\pi t\sim \\ \cos\left(\frac{1}{2}\pi t\sim^2\right)\pi t\sim \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\cos\left(\frac{1}{2}\pi t\sim^2\right)\pi^2 t\sim^2 - \sin\left(\frac{1}{2}\pi t\sim^2\right)\pi \\ -\sin\left(\frac{1}{2}\pi t\sim^2\right)\pi^2 t\sim^2 + \cos\left(\frac{1}{2}\pi t\sim^2\right)\pi \\ 0 \end{bmatrix} \tag{2}$$

Curvature

```
Norm(CrossProduct(r1, r2), 2)/Norm(r1, 2)^3 :
κ = simplify(%);
```

$$\kappa = \frac{1}{2}\pi|t\sim| \tag{3}$$

Torsion

```
DotProduct(CrossProduct(r1, r2), r3)/Norm(CrossProduct(r1, r2), 2)^2 :
τ := simplify(simplify(%));
```

$$\frac{1}{2}\pi t\sim \tag{4}$$

Tangent Vector

```
r1/Norm(r1, 2) :
T := simplify(%);
```

$$\begin{bmatrix} \frac{1}{2}\sqrt{2}\cos\left(\frac{1}{2}\pi t\sim^2\right) \\ \frac{1}{2}\sqrt{2}\sin\left(\frac{1}{2}\pi t\sim^2\right) \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \tag{5}$$

```
subs(t = 1, T) :
T1 := simplify(%);
```

$$\begin{bmatrix} 0 \\ \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{bmatrix} \tag{6}$$

Normal Vector

```
diff(T, t)/Norm(diff(T, t), 2) :
N := simplify(%);
```

$$\begin{bmatrix} -\frac{\sin\left(\frac{1}{2}\pi t\sim^2\right)t\sim}{|t\sim|} \\ \frac{\cos\left(\frac{1}{2}\pi t\sim^2\right)t\sim}{|t\sim|} \\ 0 \end{bmatrix} \tag{7}$$

$subs(t = 1, N):$
$N1 := simplify(\%);$

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Binormal Vector

$CrossProduct(T, N):$
$B := simplify(\%);$

$$\begin{bmatrix} -\frac{1}{2}\,\frac{\sqrt{2}\,\cos\left(\frac{1}{2}\,\pi\,t\sim^2\right)t\sim}{|t\sim|} \\ -\frac{1}{2}\,\frac{\sqrt{2}\,\sin\left(\frac{1}{2}\,\pi\,t\sim^2\right)t\sim}{|t\sim|} \\ \frac{1}{2}\,\frac{\sqrt{2}\,t\sim}{|t\sim|} \end{bmatrix} \tag{9}$$

$CrossProduct(T1, N1):$
$B1 := simplify(\%);$

$$\begin{bmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \tag{10}$$

Graph

$subs(t = 1, r):$
$r0 := evalf(\%);$

$$\begin{bmatrix} 0.7798934004 \\ 0.4382591474 \\ 1. \end{bmatrix} \tag{11}$$

$curve := spacecurve(r, t = -2..3, color = blue, numpoints = 1000):$
$TT := arrow(r0, T1, color = green):$
$NN := arrow(r0, N1, color = red):$
$BB := arrow(r0, B1, color = khaki):$
$display(curve, TT, NN, BB, axes = normal, scaling = constrained);$