MATHEMATICS 201-BNJ-05
Topics in Mathematics
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Monday April 27, 2008

TEST #4
SOLUTIONS

Answer all questions and show all your work. Exact answers are required.
Only the Sharp EL531 calculator is permitted.

Question 1 (10 points)

Factor $p(x)=x^4+\frac{7}{2}x^3+\frac{11}{2}x^2+6x+2$ completely over the complex numbers.

$p(x)=\frac{1}{2}\left(2x^4+7x^3+11x^2+12x+4\right)$

If $\frac{c}{d}$ is a reduced rational zero of p, then

$c\mid 4 \quad\Rightarrow\quad c=\pm1,\pm2,\pm4$

$d\mid 2 \quad\Rightarrow\quad d=\pm1,\pm2$

Thus $\frac{c}{d}=\pm\frac{1}{2},\pm1,\pm2,\pm4$

Since the coefficients are all positive, there are no positive zeros.

$$\begin{array}{r|ccccc} & 2 & 7 & 11 & 12 & 4 \\ & & -1 & -3 & -4 & -4 \\ \hline -\frac{1}{2} & 2 & 6 & 8 & 8 & 0 \end{array}$$

Thus $p(x)=\frac{1}{2}\left(x+\frac{1}{2}\right)\left(2x^3+6x^2+8x+8\right)$

$=\left(x+\frac{1}{2}\right)\left(x^3+3x^2+4x+4\right)$

We now have $\frac{c}{d}=\pm1,\pm2,\pm4$

$$\begin{array}{r|ccc} & 1 & 3 & 4 & 4 \\ & & -1 & -2 & -2 \\ \hline -1 & 1 & 2 & 2 & 2 \end{array} \qquad \begin{array}{r|ccc} & 1 & 3 & 4 & 4 \\ & & -2 & -2 & -4 \\ \hline -2 & 1 & 1 & 2 & 0 \end{array}$$

Ergo, $p(x)=\left(x+\frac{1}{2}\right)(x+2)\left(x^2+x+2\right)$

$=\left(x+\frac{1}{2}\right)(x+2)\left(x+\frac{1}{2}+\frac{\sqrt{7}}{2}i\right)\left(x+\frac{1}{2}-\frac{\sqrt{7}}{2}i\right)$

$x^2+x+2=0$

$x=\frac{-1\pm\sqrt{1-8}}{2}$

$x=\frac{-1}{2}\pm\frac{\sqrt{7}}{2}i$

Question 2 (4 points)

Find a zero (to a 6 figure accuracy) for x^3+4x+3 using Newton's method

Let $f(x)=x^3+4x+3$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x^3+4x+3}{3x^2+4}=\frac{2x^3-3}{3x^2+4}$$

$f(0)=3>0$

$f(-1)=-2<0$

$x_1=-\frac{1}{2}$

$x_2=-0.6842105263$

$x_3=-0.6735930590$

$x_4=-0.6735930853$

Thus $x\approx -0.673593$

Question 3 (4 points)

Find the x^{10}-term in the binomial expansion for $\left(2x^3-\frac{3}{4x^4}\right)^8$.

$$\left(2x^3-\frac{3}{4x^4}\right)^8=\sum_{i=0}^{8}\binom{8}{i}\left(2x^3\right)^{8-i}\left(\frac{-3}{4x^4}\right)^i$$

$$=\sum_{i=0}^{8}\binom{8}{i}2^{8-3i}x^{24-7i}(-1)^i 3^i$$

For the x^{10}-term, we have $24-7i=10$

$i=2$

Thus the x^{10}-term is $\binom{8}{2}2^{8-3\cdot 2}x^{24-7\cdot 2}(-1)^2 3^2=1008x^{10}$

Question 4 (3 points)

How many positive four digit integers are there which are even (end with 0, 2, 4, 6 or 8) and have no repetitions? (A four digit integer means that the first digit cannot be zero.)

Ends with 0: $P(9,3)=504$

Does not end with 0: $4\cdot 8\cdot P(8,2)=1792$

Thus there are 2296 possibilities

Question 5 (2 points)

The probability that Arthur hits the target with an arrow is $\frac{2}{3}$. What is the probability that he will only hit the target on his fifth shot?

$P(MMMMH)=\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{2}{3}=\frac{2}{243}$

Question 6 (6 points)

There are 16 men and 8 women in a Calculus class. A group of 5 is chosen to attend a Maple workshop.

a) What is the probability that the group will have exactly two woman and three men?

$$\frac{\binom{8}{2}\binom{16}{3}}{\binom{24}{5}} = \frac{28 \cdot 560}{42\,504} = \frac{280}{759}$$

b) What is the probability that the group will have at least one women?

$$P(\text{at least one women}) = 1 - P(\text{no women}) = 1 - \frac{\binom{16}{5}}{\binom{24}{5}} = 1 - \frac{4368}{42\,504} = \frac{227}{253}$$

Question 7 (9 points)

The probability that Mandelbrot will get a 95 on this test is 0.7, the probability that Julia will get 95 on this test is 0.4 and the probability that both will get 95 is 0.2.

a) Find the probability that Mandelbrot or Julia will get a 95 on the test.

$$\begin{aligned} P(M \cup J) &= P(M) + P(J) - P(M \cap J) \\ &= 0.7 + 0.4 - 0.2 \\ &= 0.9 \end{aligned}$$

b) Find the probability that Mandelbrot will get a 95 given that Julia has 95.

$$P(M \mid J) = \frac{P(M \cap J)}{P(J)} = \frac{0.2}{0.4} = \frac{1}{2}$$

c) Find the probability that Mandelbrot will not get 95 given that Julia did not get 95.

$$P(\bar{M} \mid \bar{J}) = \frac{P(\bar{M} \cap \bar{J})}{P(\bar{J})} = \frac{1 - P(M \cup J)}{1 - P(J)} = \frac{1 - 0.9}{1 - 0.4} = \frac{1}{6}$$

Question 8 (6 points)

The personnel at a law firm utilize three different psychological tests on ethical behavior when hiring new personnel. Exam 1 is used 60% of the time and thirty percent of those taking it fail. Exam 2 is used 30% of the time and fifty percent of those taking it fail. Exam 3 is used 10% of the time and eighty percent of those taking it fail.

a) What is the probability that an applicant fails?

$$\begin{aligned} P(F) &= P(F \text{ and } E_1) + P(F \text{ and } E_2) + P(F \text{ and } E_3) \\ &= P(E_1)P(F \mid E_1) + P(E_2)P(F \mid E_2) + P(E_3)P(F \mid E_3) \\ &= 0.6 \cdot 0.3 + 0.3 \cdot 0.5 + 0.1 \cdot 0.8 \\ &= 0.41 \end{aligned}$$

b) If an applicant fails, what is the probability that he took Exam 3?

$$\begin{aligned} P(E_3 \mid F) &= \frac{P(E_3 \text{ and } F)}{P(F)} \\ &= \frac{P(E_3)P(F \mid E_3)}{P(L)} \\ &= \frac{0.1 \cdot 0.8}{0.41} \\ &= 0.1951 \end{aligned}$$

Question 9 (6 points)

Suppose that a department store has determined that a woman who used her card last month will use it again this month with probability 0.8, and if she didn't use her card last month, then the probability she won't use it this month is 0.7. If the woman used her card this month,

a) what is the probability she will use it *n* months from now?

$$\begin{aligned} P(C_{n+1}) &= P(C_n)P(C_{n+1} \mid C_n) + P(\overline{C}_n)P(C_{n+1} \mid \overline{C}_n) \\ &= P(C_n)0.8 + (1 - P(C_n))0.3 \\ &= 0.5P(C_n) + 0.3 \end{aligned}$$

This is an affine dynamical system

Fixed point: $P = 0.5P + 0.3$

$$P = 0.6$$

Thus $P(C_n) = k(0.5)^n + 0.6$ $\qquad$ $P(C_0) = 1 = k + 0.6$

$$k = 0.4$$

Ergo, $P(C_n) = 0.4(0.5)^n + 0.6$

b) In the long run, what proportion of months will she be using the card?

$\langle P(C_n) \rangle \to 0.6$ $\qquad$ Thus she uses her card 60% of the time.