MATHEMATICS 201-BNJ-05
Topics in Mathematics
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Semester Review
SOLUTIONS

1. Find the sequence $\langle a_n \rangle_{n=0}^{\infty}$

   a) $\langle a_n \rangle$ is arithmetic with $a_3 = -2$ and $a_7 = 5$

   $$a_n = k + nd$$
   $$a_3 = k + 3d = -2 \qquad 4d = 7 \qquad a = \tfrac{-29}{4}$$
   $$a_7 = k + 7d = 5 \qquad d = \tfrac{7}{4}$$

   Thus $\langle a_n \rangle = \left\langle \tfrac{7}{4} n - \tfrac{29}{4} \right\rangle_{n=0}^{\infty}$

   b) $\langle a_n \rangle$ is geometric with $a_5 = 20$ and $a_7 = 125$

   $$a_n = ar^n$$
   $$a_5 = ar^5 = 20 \qquad r^2 = \tfrac{125}{20} = \tfrac{25}{4} \qquad a = \frac{20}{\left(\frac{25}{4}\right)^5} = \tfrac{128}{625}$$
   $$a_7 = ar^7 = 125 \qquad r = \tfrac{5}{2}$$

   Thus $\langle a_n \rangle = \left\langle \tfrac{128}{625} \left(\tfrac{5}{2}\right)^n \right\rangle_{n=0}^{\infty}$

   c) $a_n = \tfrac{2}{3}, \tfrac{-5}{9}, \tfrac{8}{27}, \tfrac{-11}{81}, \tfrac{14}{243}, \ldots$

   $$\langle a_n \rangle = \left\langle \frac{(3n+2)(-1)^n}{3^{n+1}} \right\rangle_{n=0}^{\infty}$$

   d) $\langle a_n \rangle$ is arithmetic with $a_6 = 1 - i$ and $a_9 = 4 + 2i$

   $$a_n = k + nd$$
   $$a_6 = k + 6d = 1 - i \qquad 3d = 3 + 3i \qquad k = 1 - i - 6(1+i)$$
   $$a_9 = k + 9d = 4 + 2i \qquad d = 1 + i \qquad = -5 - 7i$$

   Thus $\langle a_n \rangle = \langle -5 - 7i + (1+i)n \rangle_{n=0}^{\infty}$

e) $\langle a_n \rangle$ is geometric with $a_5 = 3+2i$ and $a_6 = 5-i$

$$a_n = ar^n$$

$$a_5 = ar^5 = 3+2i \qquad r = \tfrac{5-i}{3+2i}\cdot\tfrac{3-2i}{3-2i} \qquad a(1-i)^5 = 3+2i$$

$$a_6 = ar^6 = 5-i \qquad = \tfrac{13-13i}{13} = 1-i$$

$$a = \frac{3+2i}{\left(\sqrt{2}\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right)\right)^5}$$

$$= \frac{3+2i}{4\sqrt{2}\left(\cos\frac{-5\pi}{4}+i\sin\frac{-5\pi}{4}\right)}$$

$$= \frac{3+2i}{4\sqrt{2}\left(\frac{-\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)}$$

$$= \frac{3+2i}{-4+4i}\cdot\frac{-4-4i}{-4-4i}$$

$$= \frac{-4-20i}{32} = \frac{-1}{8} - \frac{5}{8}i$$

Thus $\langle a_n \rangle = \left\langle \left(\frac{-1}{8}-\frac{5}{8}i\right)(1-i)^n \right\rangle_{n=0}^{\infty}$

2. A rich man promises to give you \$1000 on June 1st, 2006. Each day thereafter he will give you $\frac{9}{10}$ of what he gave on the previous day. What is the first date on which the amount you receive is less than 1¢ (when the money received, rounded to the nearest cent, is less than 1)? How much have you received when this happens?

$A_0 = 1000, \qquad A_{n-1} = \frac{9}{10} A_n$

This is a geometric sequence, hence $A_n = k\left(\frac{9}{10}\right)^n$

Since $A_0 = 1000$, $\langle A_n \rangle = \left\langle 1000\left(\frac{9}{10}\right)^n \right\rangle_{n=0}^{\infty}$

Thus $1000\left(\frac{9}{10}\right)^n \le 0.005$

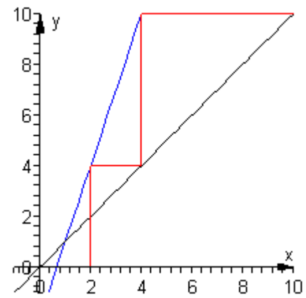
$$n \ge \frac{\ln\left(\frac{0.01}{1000}\right)}{\ln\left(\frac{9}{10}\right)} \approx 115.85$$

Hence, after 116 days.

The value of the money is $\sum_{i=0}^{116} A_i = \sum_{i=0}^{116} 1000\left(\frac{9}{10}\right)^i = \dfrac{1000\left(1-\left(\frac{9}{10}\right)^{116}\right)}{1-\frac{9}{10}} \approx 9999.95$

3. Determine if the following sequences converge. Give the exact limit of any convergent sequence. Identify all attracting, repelling and semi-attracting fixed points. For each attracting fixed point, find an interval so that if the initial value is in the interval, the sequence converges to the fixed point.

a) $a_{n+1} = 3a_n - 2 \qquad a_0 = 2$



Fixed points: $A = 3A - 2$

$A = 1$

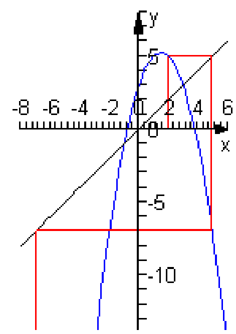
The sequence diverges for $a_0 = 2$.

$f(x) = 3x - 2$

$f'(x) = 3$

Since f' is continuous at 1 and $|f'(1)| = 3 > 1$, then 1 is a repelling fixed points.

b) $a_{n+1} = 3a_n - a_n^2 + 3 \qquad a_0 = 2$



Fixed points: $A = 3A - A^2 + 3$

$A^2 - 2A - 3 = 0$

$(A-3)(A+1) = 0$

$A = -1, 3$

The sequence diverges for $a_0 = 2$.

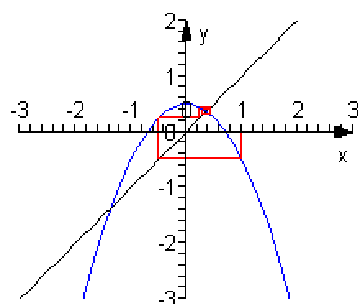
$f(x) = 3x - x^2 + 3$

$f'(x) = 3 - 2x$

Since f' is continuous at -1 and $|f'(-1)| = 5 > 1$, then -1 is a repelling fixed point.

Since f' is continuous at 3 and $|f'(3)| = 3 > 1$, then 3 is a repelling fixed point.

c) $a_{n+1} = \frac{1}{4} - a_n^2 \qquad a_0 = 1$



Fixed points: $A = \frac{1}{4} - A^2$

$0 = 4A^2 + 4A - 1$

$A = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4 \cdot (-1)}}{2 \cdot 4}$

$= -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$

For $a_0 = 1$, $\langle a_n \rangle \to \frac{-1}{2} + \frac{\sqrt{2}}{2}$.

$f(x) = \frac{1}{4} - x^2 \qquad f'(x) = -2x$

Since f' is continuous at $\frac{-1}{2} - \frac{\sqrt{2}}{2}$ and $\left|f'\left(\frac{-1}{2} - \frac{\sqrt{2}}{2}\right)\right| = 1 + \sqrt{2} > 1$, then $\frac{-1}{2} - \frac{\sqrt{2}}{2}$ is repelling

Since f' is continuous at $\frac{-1}{2} + \frac{\sqrt{2}}{2}$ and $\left|f'\left(\frac{-1}{2} + \frac{\sqrt{2}}{2}\right)\right| = \sqrt{2} - 1 < 1$, then $\frac{-1}{2} + \frac{\sqrt{2}}{2}$ is attracting. If $a_0 \in \left(\frac{-1}{2} - \frac{\sqrt{2}}{2}, \frac{1}{2} + \frac{\sqrt{2}}{2}\right)$, then $\langle a_n \rangle \to \frac{-1}{2} + \frac{\sqrt{2}}{2}$.

Note: an answer of $(-1,1)$ would also be acceptable.

d) $a_{n+1} = a_n^3 - 2a_n^2 + 2 \qquad a_0 = \frac{1}{2}$

Fixed points: $A = A^3 - 2A^2 + 2$

$$A^3 - 2A^2 - A + 2 = 0$$
$$A^2(A-2) - (A-2) = 0$$
$$(A^2 - 1)(A-2) = 0$$
$$A = -1, 1, 2$$

$f(x) = x^3 - 2x^2 + 2$

$f'(x) = 3x^2 - 4x$

Intervals of ↗ / ↘:

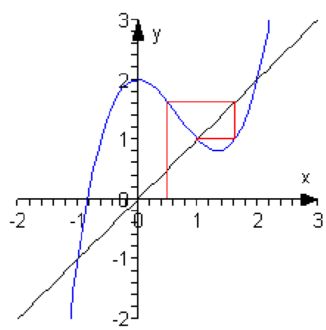
$f'(x) = 0$

$x(3x-4) = 0$

$x = 0, \frac{4}{3}$

x	$-\infty$		0		$\frac{4}{3}$		∞
f'		+	0	-	0	+	
f		↗	2 min	↘	$\frac{22}{27}$ max	↗	

For $a_0 = \frac{1}{2}$, $\langle a_n \rangle \to 1$



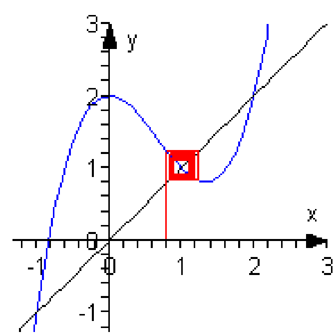
Since f' is continuous at -1 and $|f'(-1)| = 7 > 1$, then -1 is repelling.

Since f' is continuous at 1 and $|f'(1)| = 1$, the theorem is inconclusive.

For $a_0 = 0.8$ or $a_0 = 1.2$, $\langle a_n \rangle \to 1$ (although very slowly).

Thus 1 is attracting.

If $a_0 \in (-1,0) \cup (0,2)$, then $\langle a_n \rangle \to 1$



Since f' is continuous at 2 and $|f'(2)| = 4 > 1$, then 2 is repelling.

e) $a_{n+1} = a_n^3 - a_n^2 + 1 \qquad a_0 = \frac{1}{2}$

Fixed points: $A = A^3 - A^2 + 1$

$$A^3 - A^2 - A + 1 = 0$$
$$A^2(A-1) - (A-1) = 0$$
$$(A^2 - 1)(A-1) = 0$$
$$A = -1, 1$$

$f(x) = x^3 - x^2 + 1$

$f'(x) = 3x^2 - 2x$

Intervals of ↗ / ↘

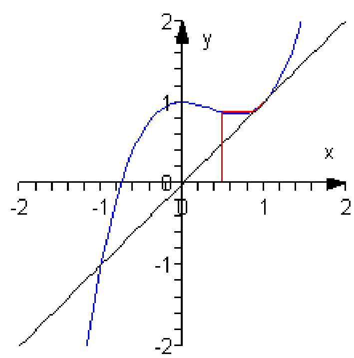
$f'(x) = 0$

$x(3x-2) = 0$

$x = 0, \frac{2}{3}$

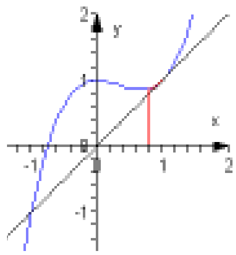
x	$-\infty$	0		$\frac{2}{3}$	∞
f'	+	0	-	0	+
f	↗	1 min	↘	$\frac{23}{27}$ max	↗

For $a_0 = \frac{1}{2}$, $\langle a_n \rangle \to 1$

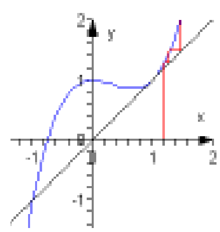


Since f' is continuous at -1 and $|f'(-1)| = 5 > 1$, then -1 is repelling.

Since f' is continuous at 1 and $|f'(1)| = 1$, the theorem is inconclusive.



For $a_0 = 0.8$ $\langle a_n \rangle \to 1$



For $a_0 = 1.2$ $\langle a_n \rangle$ diverges

Thus 1 is semi-attracting. If $a_0 \in (-1,0) \cup (0,1)$, then $\langle a_n \rangle \to 1$

4. Prove the following claims using the axiom of mathematical induction.

a) If $a_0 = 5$ and $a_{n+1} = 3a_n + 2n - 4$ then $\langle a_n \rangle = \left\langle \frac{7}{2}3^n + \frac{3}{2} - n \right\rangle_{n=0}^{\infty}$.

1. $n = 0$ $\quad LS = a_0 = 5$

$RS = \frac{7}{2}3^0 + \frac{3}{2} - 0 = 5 = LS$

2. Suppose that $a_k = \frac{7}{2}3^k + \frac{3}{2} - k$ (1)

Show that $a_{k+1} = \frac{7}{2}3^{k+1} + \frac{3}{2} - (k+1)$ (2)

LS of (2) $= a_{k+1}$

$= 3a_k + 2k - 4$

$= 3\left(\frac{7}{2}3^k + \frac{3}{2} - k\right) + 2k - 4$ by (1)

$= \frac{7}{2}3^{k+1} + \frac{9}{2} - 3k + 2k - 4$

$= \frac{7}{2}3^{k+1} + \frac{1}{2} - k$

$=$ RS of (2)

b) If $b_0 = 5$, $b_1 = 4$ and $b_{n+2} = 3b_{n+1} + 10b_n$ then $\langle b_n \rangle = \left\langle 2 \cdot 5^n + 3(-2)^n \right\rangle_{n=0}^{\infty}$

1. $n = 0$ $\quad LS = b_0 = 5$

$RS = 2 \cdot 5^0 + 3(-2)^0 = 5 = LS$

$n = 1$ $\quad LS = b_1 = 4$

$RS = 2 \cdot 5^1 + 3(-2)^1 = 4 = LS$

2. Suppose that $b_k = 2 \cdot 5^k + 3(-2)^k$ (1)

and $b_{k+1} = 2 \cdot 5^{k+1} + 3(-2)^{k+1}$ (2)

Show that $b_{k+2} = 2 \cdot 5^{k+2} + 3(-2)^{k+2}$ (3)

LS of (3) $= b_{k+2}$

$= 3b_{k+1} + 10b_k$

$= 3\left(2 \cdot 5^{k+1} + 3(-2)^{k+1}\right) + 10\left(2 \cdot 5^k + 3(-2)^k\right)$ by (1) and (2)

$= 50 \cdot 5^k + 12(-2)^k$

$= 2 \cdot 5^{k+2} + 3(-2)^{k+2}$

$=$ RS of (3)

c) $n^3 + 2n$ is divisible by 3

$n^3 + 2n$ is divisible by 3 if $n^3 + 2n = 3a$ for $a \in \mathbb{N}$

1. $n = 1$ $1^3 + 2 \cdot 1 = 3 = 3(1)$, so $n^3 + 2n$ is divisible by 3 if $n = 1$.
2. Suppose that $k^3 + 3k = 3a$, $a \in \mathbb{N}$ (1)

Show that $(k+1)^3 + 2(k+1) = 3b$, $b \in \mathbb{N}$ (2)

$$\begin{aligned}
\text{LS of (2)} &= (k+1)^3 + 2(k+1) \\
&= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= k^3 + 2k^2 + 3k^2 + 3k + 3 \\
&= 3a + 3k^2 + 3k + 3 \qquad \text{by (1)} \\
&= 3(a + k^2 + k + 1) \\
&= 3b \\
&= \text{RS of (2)}
\end{aligned}$$

d) $\sum_{i=1}^{n} i(i+1) = \frac{1}{3}n(n+1)(n+2)$

1. $n = 1$ $LS = \sum_{i=1}^{1} i(i+1) = 2$

$RS = \frac{1}{3}1(1+1)(1+2) = 2 = LS$

2. Suppose $\sum_{i=1}^{k} i(i+1) = \frac{1}{3}k(k+1)(k+2)$ (1)

Show that $\sum_{i=1}^{k+1} i(i+1) = \frac{1}{3}(k+1)(k+2)(k+3)$.

$$\begin{aligned}
\text{LS of (2)} &= \sum_{i=1}^{k+1} i(i+1) \\
&= \sum_{i=1}^{k} i(i+1) + (k+1)(k+2) \\
&= \tfrac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\
&= \tfrac{1}{3}(k+1)(k+2)(k+3) \\
&= \text{RS of (2)}
\end{aligned}$$

5. Give the following sequences in closed form.

a) $a_0 = 5 \qquad a_{n+1} = 3 - \frac{1}{2}a_n$

This is an affine system.

Fixed point: $A = 3 - \frac{1}{2}A$

$A = 2$

Thus $a_n = k\left(\frac{-1}{2}\right)^n + 2$

Since $a_0 = 5 = k + 2$ then $k = 3$

Ergo, $\langle a_n \rangle = \left\langle 3\left(\frac{-1}{2}\right)^n + 2 \right\rangle_{n=0}^{\infty}$

b) $b_0 = 3 \qquad b_{n+1} = b_n + 3n - 5$

The difference equation is $b_{n+1} - b_n = 3n - 5$

Solving the differential equation $\frac{dy}{dx} = 3x - 5$ gives us $y = \frac{3}{2}x^2 - 5x + C$

This suggests that b_n is a second degree polynomial.

Claim: $b_n = an^2 + bn + c$ for some constants *a*, *b* and *c*.

Proof:

1. $n = 0 \qquad LS = b_0 = 3 \qquad$ Thus LS = RS if $c = 3$.

$RS = c$

2. Suppose $b_k = ak^2 + bk + c \qquad (1)$

Show that $b_{k+1} = a(k+1)^2 + b(k+1) + c \qquad (2)$

$$\begin{aligned}\text{LS of (2)} &= b_{k+1}\\ &= b_k + 3k - 5\\ &= ak^2 + bk + c + 3k - 5\\ &= ak^2 + (b+3)k + c - 5\end{aligned}$$

$$\begin{aligned}\text{RS of (2)} &= a(k+1)^2 + b(k+1) + c\\ &= ak^2 + (2a+b)k + a + b + c\end{aligned}$$

Thus, if LS of (2) = RS of (2) then

$2a + b = b + 3 \qquad c - 5 = a + b + c$

$a = \frac{3}{2} \qquad b = \frac{-13}{2}$

Thus by AMI, we have $\langle b_n \rangle = \left\langle \frac{3}{2}n^2 - \frac{13}{2}n + 3 \right\rangle_{n=0}^{\infty}$

c) $a_0 = 3$, $a_1 = 4$ $\quad a_{n+2} = 2a_{n+1} + 8a_n$

This is a second order linear dynamical system.

Characteristic equation: $x^2 = 2x + 8$

$$x^2 - 2x - 8 = 0$$
$$(x-4)(x+2) = 0$$
$$x = -2, 4$$

Thus $a_n = k(-2)^n + l4^n$

$a_0 = 3 = k + l$ $\quad 10 = 6l$ $\quad k = \frac{4}{3}$

$a_1 = 4 = -2k + 4l$ $\quad l = \frac{5}{3}$

Ergo, $\langle a_n \rangle = \left\langle \frac{4}{3}(-2)^n + \frac{5}{3}4^n \right\rangle_{n=0}^{\infty}$

d) $b_0 = 1$, $b_1 = 4$ $\quad a_{n+2} = 4a_{n+1} - 13a_n + 20$

This is a second order affine dynamical system.

Fixed point: $A = 4A - 13A + 20$

$A = 2$

Characteristic equation: $x^2 = 4x - 13$

$$x^2 - 4x + 13 = 0$$
$$x = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 13}}{2}$$
$$x = 2 \pm 3i$$

Thus $b_n = k(2+3i)^n + l(2-3i)^n + 2$

$b_0 = 1 = k + l + 2$ $\quad -2 - 3i - 2 = 6il$ $\quad k = \frac{-1}{2} - \frac{2}{3}i$

$b_1 = 4 = k(2+3i) + l(2-3i) + 2$ $\quad l = \frac{-4-3i}{6i}$

$= \frac{-1}{2} + \frac{2}{3}i$

Ergo, $\langle b_n \rangle = \left\langle \left(\frac{-1}{2} - \frac{2}{3}i\right)(2+3i)^n + \left(\frac{-1}{2} + \frac{2}{3}i\right)(2-3i)^n + 2 \right\rangle_{n=0}^{\infty}$

e) $c_0 = -2$, $c_1 = 10$ $\quad c_{n+2} = 10c_{n+1} - 25c_n + 16$

This is a second order affine dynamical system.

Fixed point: $A = 10A - 25A + 16$ $\quad$ Characteristic equation: $x^2 = 10x - 25$

$A = 1$

$$x^2 - 10x + 25 = 0$$
$$(x-5)^2 = 0$$
$$x = 5$$

Thus $c_n = (k + ln)5^n + 1$

$c_0 = -2 = k + 1$ $\quad k = -3$

$c_1 = 10 = (k + l)5 + 1$ $\quad l = \frac{24}{5}$

Ergo, $\langle c_n \rangle = \left\langle \left(-3 + \frac{24}{5}n\right)5^n + 1 \right\rangle_{n=0}^{\infty}$

6. Use basic sigma properties to show $\sum_{i=2}^{n+1}\left((i-1)^2-2i+6\right)=\sum_{i=0}^{n}\left(i^2+1\right)-(n-1)^2$ without expanding.

$$\begin{aligned}\sum_{i=2}^{n+1}\left((i-1)^2-2i+6\right)&=\sum_{i=0}^{n-1}\left(((i+2)-1)^2-2(i+2)+6\right)\\&=\sum_{i=0}^{n-1}\left(i^2+2i+1-2i-2+6\right)\\&=\sum_{i=0}^{n-1}\left(i^2+1\right)+\sum_{i=0}^{n-1}2\\&=\sum_{i=0}^{n-1}\left(i^2+1\right)+\sum_{i=n}^{n}\left(i^2+1\right)-\sum_{i=n}^{n}\left(i^2+1\right)+2n\\&=\sum_{i=0}^{n}\left(i^2+1\right)-\left(n^2+1\right)+2n\\&=\sum_{i=0}^{n}\left(i^2+1\right)-n^2+2n-1\\&=\sum_{i=0}^{n}\left(i^2+1\right)-(n-1)^2\end{aligned}$$

7. Evaluate $\sum_{i=1}^{3}\sum_{j=-26}^{27} i\sqrt[3]{j}$.

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=-26}^{27} i\sqrt[3]{j}&=\sum_{i=1}^{3} i\sum_{j=-26}^{27}\sqrt[3]{j}\\&=(1+2+3)\left(\sum_{j=-26}^{0}\sqrt[3]{j}+\sum_{j=1}^{27}\sqrt[3]{j}\right)\\&=6\left(\sum_{j=0}^{26}\sqrt[3]{-j}+\sum_{j=1}^{27}\sqrt[3]{j}\right)\\&=6\left(-\sum_{j=1}^{26}\sqrt[3]{j}+\sum_{j=1}^{26}\sqrt[3]{j}+\sqrt[3]{27}\right)\\&=18\end{aligned}$$

8. Find the sum of the first n terms of $\left\langle 2n-5^n+3\right\rangle_{n=0}^{\infty}$.

$$\begin{aligned}\sum_{i=0}^{n-1}\left(2i-5^i+3\right)&=-\sum_{i=0}^{n-1}5^i+\sum_{i=0}^{n-1}(2i+3)\\&=-\frac{\left(1-5^n\right)}{1-5}+n\frac{2(n-1)+3+3}{2} \qquad \text{(geometric and arithmetic series)}\\&=\tfrac{1}{4}-\tfrac{1}{4}5^n+n^2+2n\\&=\tfrac{-1}{4}5^n+n^2+2n+\tfrac{1}{4}\end{aligned}$$

9. Find the exact value of the following sums.

a) $\sum_{n=2}^{20}(3n-5)=(20-2+1)\left(\frac{(3(20)-5)+(3(2)-5)}{2}\right)=532$ (arithmetic series)

b) $\sum_{n=0}^{6}\frac{4\cdot 5^{n+1}}{3\cdot 2^{3n-1}}=\frac{40}{3}+\frac{25}{3}+\cdots+\frac{4\cdot 5^{7}}{3\cdot 2^{15}}=\sum_{n=0}^{6}\left(\frac{40}{3}\right)\left(\frac{5}{8}\right)^{n}$

$=\frac{\frac{40}{3}\left(1-\left(\frac{5}{8}\right)^{7}\right)}{1-\frac{5}{8}}=\frac{3365045}{98304}$ (geometric series)

c) $\sum_{n=0}^{10}(2+3i-4n+2ni)=(10+1)\left(\frac{(2+3i)+(-38+23i)}{2}\right)$ (arithmetic series)

$=-198+143i$

d) $\sum_{n=0}^{6}(1+2i)(1+i)^{n}=\frac{(1+2i)\left(1-(1+i)^{7}\right)}{1-(1+i)}$ (geometric series)

$=\frac{(1+2i)\left(1-\left(\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)^{7}\right)}{-i}$

$=\frac{(1+2i)(1-(8-8i))}{-i}\cdot\frac{i}{i}$

$=(-2+i)(-7+8i)$

$=6-23i$

10. Write the infinite decimal expansion of $3.1\overline{415}$ as quotient of two integers.

$3.1\overline{415}=3.1+0.0415+0.0000415+\ldots$

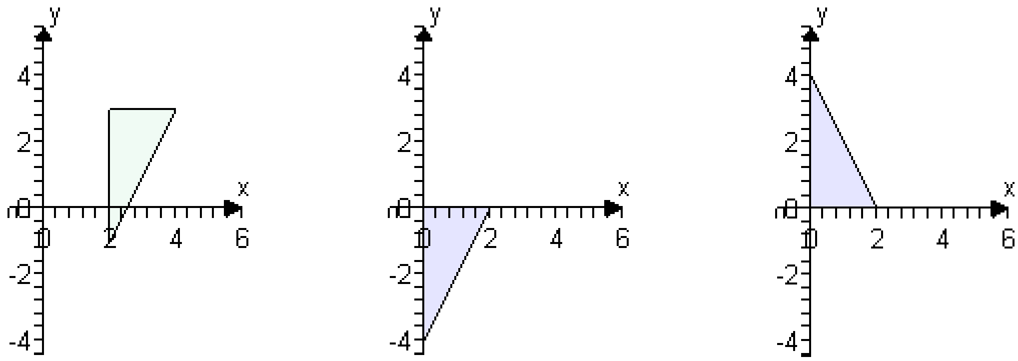
$=\frac{31}{10}+\sum_{n=0}^{\infty}\frac{415}{10000}\left(\frac{1}{1000}\right)^{n}$

$=\frac{31}{10}+\frac{\frac{415}{10000}}{1-\frac{1}{1000}}$

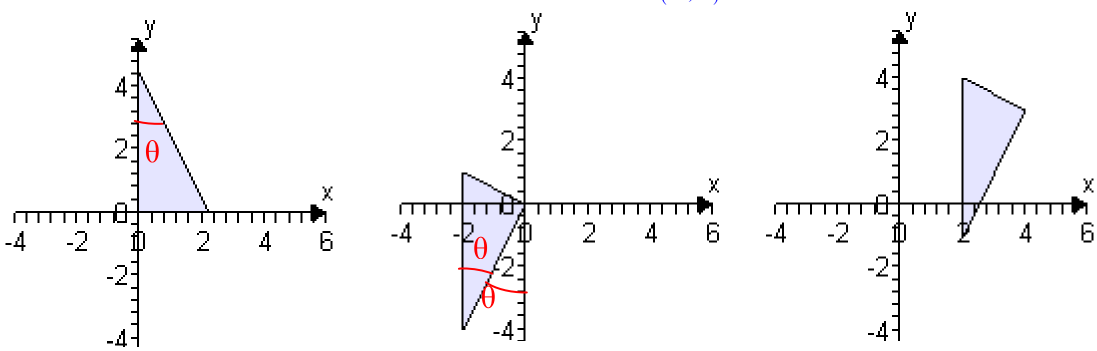
$=\frac{31}{10}+\frac{415}{9990}$

$=\frac{15692}{4995}$

11. Find the planar transformation F that maps triangle ABC with $A=(2,-1)$, $B=(2,3)$ and $C=(4,3)$ to triangle $A'B'C'$ where $A'=f(A)=A$, $B'=f(B)=C$ and $C'=f(C)=(2,4)$.



Start with translation $T_{(-2,-3)}$ Reflection M_x



Dilation: $\frac{\|AC\|}{\|AB\|}=\frac{2\sqrt{5}}{4}$

Thus $D_{\frac{\sqrt{5}}{2}}$

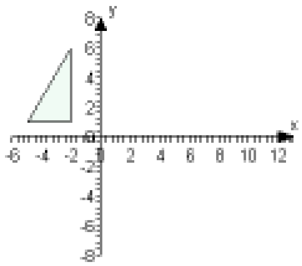
Rotation R_β where

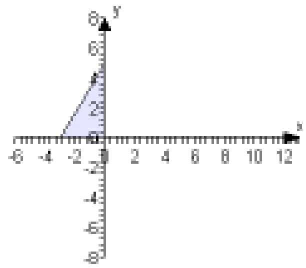
$$\beta=\pi-\theta$$
$$=\pi-\arctan\tfrac{1}{2}$$
$$\cos\beta=\tfrac{-2}{\sqrt{5}},\quad \sin\beta=\tfrac{1}{\sqrt{5}}$$

Translation $T_{(4,3)}$

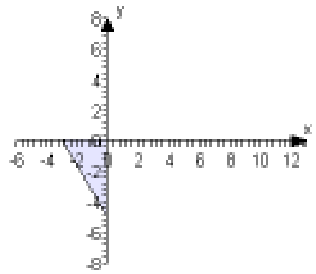
$$\begin{aligned}
F(x,y)&=T_{(4,3)}\circ R_\beta\circ D_{\frac{\sqrt{5}}{2}}\circ M_x\circ T_{(-2,-3)}(x,y)\\
&=T_{(4,3)}\circ R_\beta\circ D_{\frac{\sqrt{5}}{2}}\left(x-2,-(y-3)\right)\\
&=T_{(4,3)}\circ R_\beta\left(\tfrac{\sqrt{5}}{2}x-\sqrt{5},-\tfrac{\sqrt{5}}{2}y+\tfrac{3\sqrt{5}}{2}\right)\\
&=T_{(4,3)}\left(\left(\tfrac{\sqrt{5}}{2}x-\sqrt{5}\right)\tfrac{-2}{\sqrt{5}}-\left(-\tfrac{\sqrt{5}}{2}y+\tfrac{3\sqrt{5}}{2}\right)\tfrac{1}{\sqrt{5}},\tfrac{1}{\sqrt{5}}\left(\tfrac{\sqrt{5}}{2}x-\sqrt{5}\right)-\tfrac{2}{\sqrt{5}}\left(-\tfrac{\sqrt{5}}{2}y+\tfrac{3\sqrt{5}}{2}\right)\right)\\
&=\left(-x+2+\tfrac{1}{2}y-\tfrac{3}{2}+4,\tfrac{1}{2}x-1+y-3+3\right)\\
&=\left(-x+\tfrac{1}{2}y+\tfrac{9}{2},\tfrac{1}{2}x+y-1\right)
\end{aligned}$$

12. Find the planar transformation F that maps triangle ABC with $A=(-2,1)$, $B=(-5,1)$ and $C=(-2,6)$ to triangle $A'B'C'$ where $A'=f(A)=(2,4)$, $B'=f(B)=(2,-2)$ and $C'=f(C)=(12,4)$.

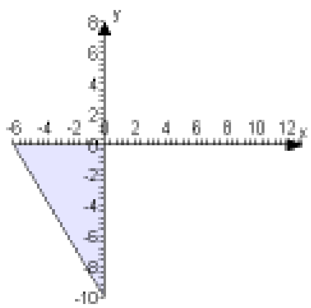




Start with translation $T_{(2,-1)}$

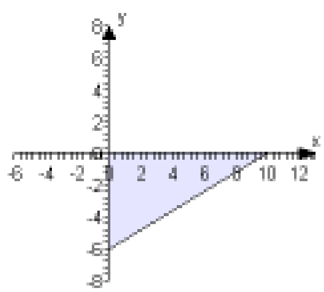


Reflection M_x

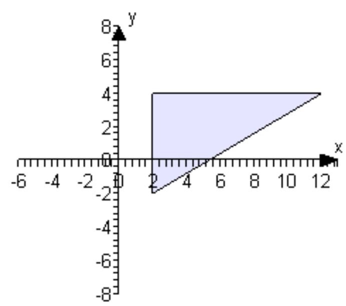


Dilation: $\frac{\|A'B'\|}{\|AC\|}=\frac{10}{5}$

Thus D_2



Rotation $R_{\frac{\pi}{2}}$



Translation $T_{(2,4)}$

$$\begin{aligned} F(x,y) &= T_{(2,4)}\circ R_{\frac{\pi}{2}}\circ D_2\circ M_x\circ T_{(2,-1)}(x,y) \\ &= T_{(2,4)}\circ R_{\frac{\pi}{2}}\circ D_2(x+2,-y+1) \\ &= T_{(2,4)}\circ R_{\frac{\pi}{2}}(2x+4,-2y+2) \\ &= T_{(2,4)}(-(-2y+2),(2x+4)) \\ &= (2y-2+2,2x+4+4) \\ &= (2y,2x+8) \end{aligned}$$

13. Find the image of the line $y=4x+3$ under the planar transformation $F=D_2\circ M_x\circ T_{(-3,1)}$.

Let (x,y) be any point on the image of the line.

Then $(u,v)=F^{-1}(x,y)$

$$\begin{aligned} &= T_{(3,-1)}\circ M_x\circ D_{\frac{1}{2}}(x,y) \\ &= \left(\tfrac{1}{2}x+3,\tfrac{-1}{2}y-1\right) \end{aligned}$$

is a point on the given line. Thus

$$\begin{aligned} v &= 4u+3 \\ \tfrac{-1}{2}y-1 &= 4\left(\tfrac{1}{2}x+3\right)+3 \\ y &= -4x-32 \end{aligned}$$

14. If $\langle (x_n, y_n) \rangle$ is defined recursively by $(x_0, y_0) = (4,1)$ and $(x_{n+1}, y_{n+1}) = F(x_n, y_n)$, then find (x_n, y_n) in closed form.

a) $F = T_{(2,-1)} \circ D_{\frac{2}{3}}$

$(x_{n+1}, y_{n+1}) = \left(\frac{2}{3}x_n + 2, \frac{2}{3}y_n - 1\right)$

$x_{n+1} = \frac{2}{3}x_n + 2$ and $y_{n+1} = \frac{2}{3}y_n - 1$

These are two affine dynamical systems,

Fixed points: $X = \frac{2}{3}X + 2$ $\quad$ $Y = \frac{2}{3}Y - 1$

$X = 6$ $\quad$ $Y = -3$

Thus $x_n = k\left(\frac{2}{3}\right)^n + 6$ $\quad$ $y_n = l\left(\frac{2}{3}\right)^n - 3$

$x_0 = 4 = k + 6$ $\quad$ $y_0 = 1 = l - 3$

$k = -2$ $\quad$ $l = 4$

Ergo, $\langle (x_n, y_n) \rangle = \left\langle -2\left(\frac{2}{3}\right)^n + 6, 4\left(\frac{2}{3}\right)^n - 3 \right\rangle_{n=0}^{\infty}$

b) $F(x,y) = (3x - y, 4x + 7y)$

$x_{n+1} = 3x_n - y_n$ $\quad$ $y_{n+1} = 4x_n + 7y_n$ (3)

$y_n = -x_{n+1} + 3x_n$ (1)

$y_{n+1} = -x_{n+2} + 3x_{n+1}$ (2)

Combining (1) and (2) into (3), $-x_{n+2} + 3x_{n+1} = 4x_n + 7(-x_{n+1} + 3x_n)$

$x_{n+2} = 10x_{n+1} - 25x_n$

This is a second order linear dynamical system.

Characteristic equation: $x^2 = 10x - 25$

$x^2 - 10x + 25 = 0$

$(x-5)^2 = 0$

$x = 5$

Thus $x_n = (k + l\,n)5^n$

From (1) we obtain

$y_n = -(k + l(n+1))5^{n+1} + 3(k + l\,n)5^n$

$= (-2k + 5l - 2l\,n)5^n$

We have $x_0 = 4 = k$

$y_0 = 1 = -2k - 5l$ $\quad$ $l = \frac{-9}{5}$

Thus $\langle (x_n, y_n) \rangle = \left\langle \left(4 - \frac{9}{5}n\right)5^n, \left(1 + \frac{18}{5}n\right)5^n \right\rangle_{n=0}^{\infty}$

c) $F(x,y)=(3x-5y+1,-3x+y+2)$

$$x_{n+1}=3x_n-5y_n+1 \qquad y_{n+1}=-3x_n+y_n+2 \quad (3)$$

$$y_n=-\tfrac{1}{5}x_{n+1}+\tfrac{3}{5}x_n+\tfrac{1}{5} \quad (1)$$

$$y_{n+1}=-\tfrac{1}{5}x_{n+2}+\tfrac{3}{5}x_{n+1}+\tfrac{1}{5} \quad (2)$$

Combining (1) and (2) into (3), $-\tfrac{1}{5}x_{n+2}+\tfrac{3}{5}x_{n+1}+\tfrac{1}{5}=-3x_n-\tfrac{1}{5}x_{n+1}+\tfrac{3}{5}x_n+\tfrac{1}{5}+2$

$$x_{n+2}=4x_{n+1}+12x_n-10$$

This is a second order affine dynamical system.

Fixed point: $X=4X+12X-10$

$$X=\tfrac{2}{3}$$

Characteristic equation: $x^2=4x+12$

$$x^2-4x-12=0$$

$$(x-6)(x+2)=0$$

$$x=-2,6$$

Thus $x_n=k(-2)^n+l(6)^n+\tfrac{2}{3}$

From (1) we obtain

$$y_n=-\tfrac{1}{5}\left(k(-2)^{n+1}+l(6)^{n+1}+\tfrac{2}{3}\right)+\tfrac{3}{5}\left(k(-2)^n+l(6)^n+\tfrac{2}{3}\right)+\tfrac{1}{5}$$

$$=k(-2)^n-\tfrac{3}{5}(6)^n+\tfrac{7}{15}$$

We have $x_0=4=k+l+\tfrac{2}{3}$ $\qquad 3=\tfrac{8}{5}l+\tfrac{1}{5}$ $\qquad k=\tfrac{10}{3}-\tfrac{7}{4}=\tfrac{19}{12}$

$y_0=1=k-\tfrac{3}{5}l+\tfrac{7}{15}$ $\qquad l=\tfrac{7}{4}$

Thus $\langle(x_n,y_n)\rangle=\left\langle\left(\tfrac{19}{12}(-2)^n+\tfrac{7}{4}6^n+\tfrac{2}{3},\tfrac{19}{12}(-2)^n-\tfrac{21}{20}6^n+\tfrac{7}{15}\right)\right\rangle_{n=0}^{\infty}$

d) $F=T_{(1,-1)}\circ D_{\frac{1}{2}}\circ R_{\frac{\pi}{3}}$

In the complex plane,

$$z_{n+1}=F(z_n)=\tfrac{1}{2}\left(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3}\right)z_n+1-i$$

$$=\left(\tfrac{1}{4}+\tfrac{\sqrt{3}}{4}i\right)z_n+1-i$$

This is an affine system,

Fixed Point:

$$Z=\left(\tfrac{1}{4}+\tfrac{\sqrt{3}}{4}i\right)Z+1-i$$

$$\left(\tfrac{3}{4}-\tfrac{\sqrt{3}}{4}i\right)Z=1-i$$

$$Z=\frac{4-4i}{3-\sqrt{3}i}\,\frac{3+\sqrt{3}i}{3+\sqrt{3}i}=\frac{12+4\sqrt{3}+\left(-12+4\sqrt{3}\right)i}{12}=\tfrac{3+\sqrt{3}}{3}+\tfrac{-3+\sqrt{3}}{3}i$$

Hence $z_n = k\left(\frac{1}{4}+\frac{\sqrt{3}}{4}\right)^n + \frac{3+\sqrt{3}}{3} + \frac{-3+\sqrt{3}}{3}i$

$$z_0 = 4+i = k + \frac{3+\sqrt{3}}{3} + \frac{-3+\sqrt{3}}{3}i$$

$$k = \frac{9-\sqrt{3}}{3} + \frac{6-\sqrt{3}}{3}i$$

Which gives us,

$$z_n = \left(\frac{9-\sqrt{3}}{3}+\frac{6-\sqrt{3}}{3}i\right)\left(\frac{1}{4}+\frac{\sqrt{3}}{4}i\right)^n + \frac{3+\sqrt{3}}{3} + \frac{-3+\sqrt{3}}{3}i$$

$$= \left(\frac{9-\sqrt{3}}{3}+\frac{6-\sqrt{3}}{3}i\right)\left(\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right)^n + \frac{3+\sqrt{3}}{3} + \frac{-3+\sqrt{3}}{3}i$$

$$= \left(\frac{9-\sqrt{3}}{3}\cos\frac{n\pi}{3}-\frac{6-\sqrt{3}}{3}\sin\frac{n\pi}{3}\right)\left(\frac{1}{2}\right)^n + \frac{3+\sqrt{3}}{3} + \left(\frac{9-\sqrt{3}}{3}\sin\frac{n\pi}{3}+\frac{6-\sqrt{3}}{3}\cos\frac{n\pi}{3}\right)\left(\frac{1}{2}\right)^n + \frac{-3+\sqrt{3}}{3}i$$

Ergo,

$$\langle(x_n, y_n)\rangle = \left\langle\left(\left(\frac{9-\sqrt{3}}{3}\cos\frac{n\pi}{3}-\frac{6-\sqrt{3}}{3}\sin\frac{n\pi}{3}\right)\left(\frac{1}{2}\right)^n + \frac{3+\sqrt{3}}{3}, \left(\frac{9-\sqrt{3}}{3}\sin\frac{n\pi}{3}+\frac{6-\sqrt{3}}{3}\cos\frac{n\pi}{3}\right)\left(\frac{1}{2}\right)^n + \frac{-3+\sqrt{3}}{3}i\right)\right\rangle_{n=0}^{\infty}$$

15. Consider the following fractal A given below.

a) Give a recursive definition for this fractal.

$$f_1(x,y) = D_{\frac{1}{3}}(x,y) = \left(\tfrac{1}{3}x, \tfrac{1}{3}y\right)$$

$$f_2(x,y) = T_{\left(\frac{1}{3},0\right)} \circ D_{\frac{1}{3}}(x,y) = \left(\tfrac{1}{3}x+\tfrac{1}{3}, \tfrac{1}{3}y\right)$$

$$f_3(x,y) = T_{\left(\frac{2}{3},0\right)} \circ D_{\frac{1}{3}}(x,y) = \left(\tfrac{1}{3}x+\tfrac{2}{3}, \tfrac{1}{3}y\right)$$

$$f_4(x,y) = T_{\left(\frac{1}{6},\frac{1}{3}\right)} \circ D_{\frac{2}{3}}(x,y) = \left(\tfrac{2}{3}x+\tfrac{1}{6}, \tfrac{2}{3}y+\tfrac{1}{3}\right)$$

Thus A_0 is the filled triangle with vertices $(0,0)$, $(2,0)$ and $(0,1)$

and $A_{n+1} = f_1(A_n) \cup f_2(A_n) \cup f_3(A_n) \cup f_4(A_n)$

b) How would you find the dimension?

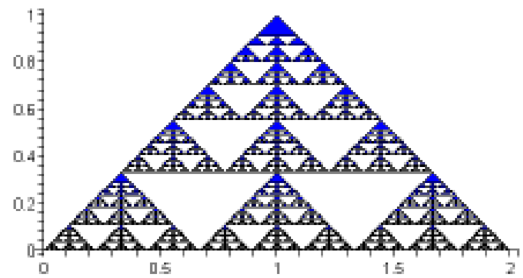
Solve the equation $\left(\frac{1}{3}\right)^D + \left(\frac{1}{3}\right)^D + \left(\frac{1}{3}\right)^D + \left(\frac{2}{3}\right)^D = 1$ for D, the dimension.

c) How could make a sequence of points (not sets) in the plane having A as an attractor?

$$p_i = \frac{s_i^2}{\sum_{i=1}^{4} s_i^2} = \frac{s_i^2}{\frac{7}{9}}$$

Thus $p_1 = \frac{\left(\frac{1}{3}\right)^2}{\frac{7}{9}} = \frac{1}{7}$, $p_2 = \frac{\left(\frac{1}{3}\right)^2}{\frac{7}{9}} = \frac{1}{7}$, $p_3 = \frac{\left(\frac{1}{3}\right)^2}{\frac{7}{9}} = \frac{1}{7}$, $p_4 = \frac{\left(\frac{2}{3}\right)^2}{\frac{7}{9}} = \frac{4}{7}$

Sequence of points $\langle a_n\rangle$ is defined by $a_0 = (0,0)$, and $a_{n+1} = f_i(a_n)$ where f_i is chosen among the 4 defining functions with probability p_i, where $p_1 = p_2 = p_3 = \frac{1}{7}$ and $p_4 = \frac{4}{7}$.



16. Consider the following fractal A, constructed as shown below.

a) Give a recursive definition for this fractal.

$$f_1(x,y)=D_{\frac{\sqrt{2}}{4}}\circ R_{\frac{\pi}{4}}(x,y)=\left(\tfrac{1}{4}x-\tfrac{1}{4}y,\tfrac{1}{4}x+\tfrac{1}{4}y\right)$$

$$f_2(x,y)=T_{\left(\frac{1}{4},\frac{1}{4}\right)}\circ D_{\frac{\sqrt{2}}{4}}\circ R_{-\frac{\pi}{4}}(x,y)=\left(\tfrac{1}{4}x+\tfrac{1}{4}y+\tfrac{1}{4},-\tfrac{1}{4}x+\tfrac{1}{4}y+\tfrac{1}{4}\right)$$

$$f_3(x,y)=T_{\left(\frac{1}{2},0\right)}\circ D_{\frac{\sqrt{2}}{4}}\circ R_{-\frac{\pi}{4}}(x,y)=\left(\tfrac{1}{4}x+\tfrac{1}{4}y+\tfrac{1}{2},-\tfrac{1}{4}x+\tfrac{1}{4}y\right)$$

$$f_4(x,y)=T_{\left(\frac{3}{4},-\frac{1}{4}\right)}\circ D_{\frac{\sqrt{2}}{4}}\circ R_{\frac{\pi}{4}}(x,y)=\left(\tfrac{1}{4}x-\tfrac{1}{4}y+\tfrac{3}{4},\tfrac{1}{4}x+\tfrac{1}{4}y-\tfrac{1}{4}\right)$$

Thus $A_0=\{(t,0)\mid 0\le t\le 1\}$ and $A_{n+1}=f_1(A_n)\cup f_2(A_n)\cup f_3(A_n)\cup f_4(A_n)$

b) Find the dimension.

$$\left(\tfrac{\sqrt{2}}{4}\right)^D+\left(\tfrac{\sqrt{2}}{4}\right)^D+\left(\tfrac{\sqrt{2}}{4}\right)^D+\left(\tfrac{\sqrt{2}}{4}\right)^D=1$$

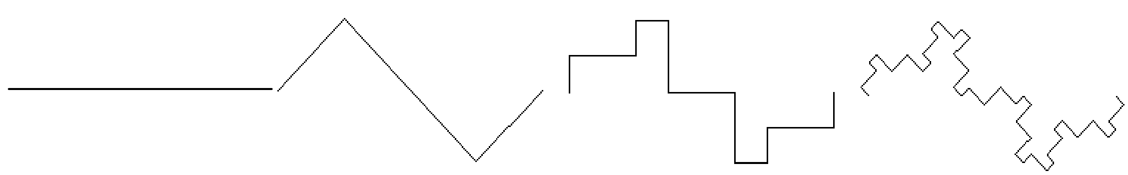
$$\left(\tfrac{\sqrt{2}}{4}\right)^D=\tfrac{1}{4}$$

$$D=\frac{\ln\frac{1}{4}}{\ln\frac{\sqrt{2}}{4}}=\frac{4}{3}$$

c) How could make a sequence of points (not sets) in the plane having A as an attractor?

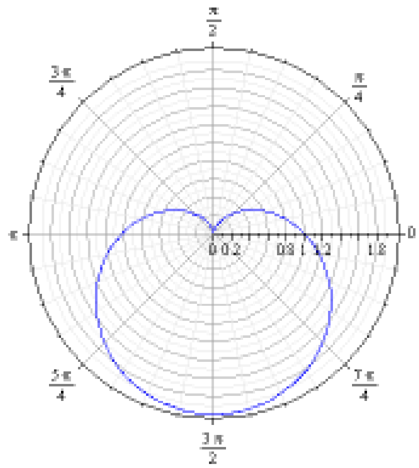
$$p_i=\frac{s_i^2}{\sum_{i=1}^{4}s_i^2}=\frac{\left(\frac{\sqrt{2}}{4}\right)^2}{4\left(\frac{\sqrt{2}}{4}\right)^2}=\frac{1}{4}$$

Sequence of points $\langle a_n\rangle$ is defined by $a_0=(0,0)$, and $a_{n+1}=f_i(a_n)$ where f_i is chosen among the 4 defining functions each with probability $p_i=0.25$.

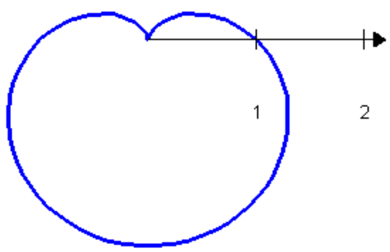


17. Sketch the graphs of the polar equations, and transform the equations into Cartesian form.

a) $r = 1 - \sin\theta$

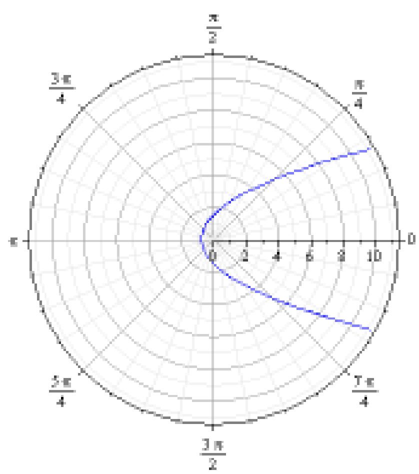


θ	$r = 1 - \sin\theta$
0	1
$\frac{\pi}{4}$	$\frac{2-\sqrt{2}}{2}$
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	$\frac{2-\sqrt{2}}{2}$
π	1
$\frac{5\pi}{4}$	$\frac{2+\sqrt{2}}{2}$
$\frac{3\pi}{2}$	2
$\frac{7\pi}{4}$	$\frac{2+\sqrt{2}}{2}$
2π	1

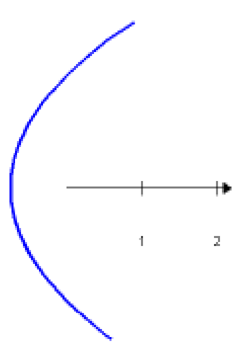


$$r^2 = r - r\sin\theta$$
$$x^2 + y^2 = \sqrt{x^2 + y^2} - y$$
$$x^2 + y^2 + y = \sqrt{x^2 + y^2}$$
$$\left(x^2 + y^2 + y\right)^2 = x^2 + y^2$$

b) $r = \frac{3}{2-2\cos\theta}$



θ	$r = 1 - \sin\theta$
0	$\nexists$
$\frac{\pi}{4}$	$\frac{3}{2-\sqrt{2}}$
$\frac{\pi}{2}$	$\frac{3}{2}$
$\frac{3\pi}{4}$	$\frac{3}{2+\sqrt{2}}$
π	$\frac{3}{4}$
$\frac{5\pi}{4}$	$\frac{3}{2+\sqrt{2}}$
$\frac{3\pi}{2}$	$\frac{3}{2}$
$\frac{7\pi}{4}$	$\frac{3}{2-\sqrt{2}}$
2π	$\nexists$



$$2r - 2r\cos\theta = 3$$
$$2\sqrt{x^2 + y^2} - 2x = 3$$
$$2\sqrt{x^2 + y^2} = 2x + 3$$
$$4\left(x^2 + y^2\right) = (2x + 3)^2$$
$$4x^2 + 4y^2 = 4x^2 + 12x + 9$$
$$x = \tfrac{1}{3}y^2 - \tfrac{3}{4}$$

18. Give the following numbers in standard form.

a) $\dfrac{2-i}{i(3+4i)}=\dfrac{2-i}{-4+3i}\cdot\dfrac{-4-3i}{-4-3i}=\dfrac{-11-2i}{25}=\frac{-11}{25}-\frac{2}{25}i$

b) $\dfrac{(1-i)(2+i)}{(1+i)(2-i)}=\dfrac{3-i}{3+i}\cdot\dfrac{3-i}{3-i}=\dfrac{8-6i}{10}=\frac{4}{5}-\frac{3}{5}i$

c) $\sqrt[7]{-2-2\sqrt{3}i}$ $\quad\left|-2-2\sqrt{3}i\right|=\sqrt{16}=4$ $\quad\arg\left(-2-2\sqrt{3}i\right)=\frac{-2\pi}{3}$

$$-2-2\sqrt{3}i=4\left(\cos\tfrac{-2\pi}{3}+i\sin\tfrac{-2\pi}{3}\right)$$ \

$$\left(-2-2\sqrt{3}i\right)^{\frac{1}{7}}=\sqrt[7]{4}\left(\cos\frac{\frac{-2\pi}{3}+2\pi k}{7}+i\sin\frac{\frac{-2\pi}{3}+2\pi k}{7}\right)$$

$$=\sqrt[7]{4}\left(\cos\tfrac{2\pi(3k-1)}{21}+i\sin\tfrac{2\pi(3k-1)}{21}\right)\qquad k=0,1,2,3,4,5,6$$

Thus $\sqrt[7]{-2-2\sqrt{3}i}=\sqrt[7]{4}\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=-\frac{\sqrt[7]{4}}{2}-\frac{\sqrt[7]{4}\sqrt{3}}{2}i$ (taking $k=5$)

d) $\sqrt{e^{2-2i}}$

$$e^{2-2i}=e^2e^{-2i}=e^2\left(\cos(-2)+i\sin(-2)\right)$$

$$\left(e^{2-2i}\right)^{\frac{1}{2}}=\sqrt{e^2}\left(\cos\tfrac{-2+2\pi k}{2}+i\sin\tfrac{-2+2\pi k}{2}\right)$$

$$=\sqrt{e^2}\left(\cos(k\pi-1)+i\sin(k\pi-1)\right)\qquad k=0,1$$

Thus $\sqrt{e^{2-2i}}=\sqrt{e^2}\left(\cos(-1)+i\sin(-1)\right)$

$$=e\left(\cos(-1)+i\sin(-1)\right)$$

$$=e^{1-i}$$

e) $\arccos\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-i\ln\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i+\sqrt{\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^2-1}\right)$

$$=-i\ln\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i+\sqrt{\tfrac{-1}{2}+\tfrac{\sqrt{3}}{2}i}\right)\qquad\left|\tfrac{-1}{2}+\tfrac{\sqrt{3}}{2}i\right|=1$$

$$=-i\ln\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i+\left(\cos\left(\tfrac{2\pi}{3}\right)+i\sin\left(\tfrac{2\pi}{3}\right)\right)^{\frac{1}{2}}\right)\qquad\operatorname{Arg}\left(\tfrac{-1}{2}+\tfrac{\sqrt{3}}{2}i\right)=\tfrac{2\pi}{3}$$

$$=-i\ln\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i+\cos\left(\tfrac{\pi}{3}\right)+i\sin\left(\tfrac{\pi}{3}\right)\right)$$

$$=-i\ln\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}+\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}\right)i\right)$$

$$=-i\ln\left(\left(\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{6}}{2}\right)\left(\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}\right)\right)\qquad\left|\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}+\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}\right)i\right|=\sqrt{2}\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}\right)$$

$$=-i\left(\ln\left|\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{6}}{2}\right|+i\tfrac{\pi}{4}\right)\qquad\operatorname{Arg}\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}+\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}\right)i\right)=\tfrac{\pi}{4}$$

$$=\tfrac{\pi}{4}-i\ln\left(\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{6}}{2}\right)$$

f) $$\operatorname{arctanh}\left(2+\sqrt{3}i\right)=\tfrac{1}{2}\ln\left(\frac{1+2+\sqrt{3}i}{1-2-\sqrt{3}i}\right)$$
$$=\tfrac{1}{2}\ln\left(\frac{3+\sqrt{3}i}{-1-\sqrt{3}i}\,\frac{-1+\sqrt{3}i}{-1+\sqrt{3}i}\right)$$
$$=\tfrac{1}{2}\ln\left(\tfrac{-3}{2}+\tfrac{\sqrt{3}}{2}i\right)$$
$$=\tfrac{1}{2}\left(\ln\sqrt{3}+i\tfrac{5\pi}{6}\right)$$
$$=\tfrac{1}{4}\ln 3+\tfrac{5\pi}{12}i$$

$$\left|\tfrac{-3}{2}+\tfrac{\sqrt{3}}{2}i\right|=\sqrt{3}$$
$$\operatorname{Arg}\left(\tfrac{-3}{2}+\tfrac{\sqrt{3}}{2}i\right)=\tfrac{5\pi}{6}$$

19. Give the following numbers in trigonometric and in exponential form.

a) $$\left(2+2i\right)^7=\left(2\sqrt{2}\left(\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}\right)\right)^7$$
$$=1024\sqrt{2}\left(\cos\tfrac{7\pi}{4}+i\sin\tfrac{7\pi}{4}\right)$$
$$=1024\sqrt{2}\left(\cos\left(\tfrac{-\pi}{4}\right)+i\sin\left(\tfrac{-\pi}{4}\right)\right)$$
$$=1024\sqrt{2}e^{\frac{-\pi}{4}i}$$

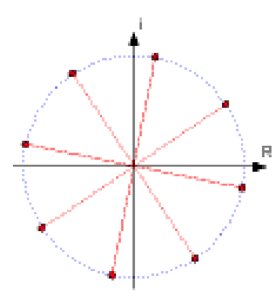
$$\left|2+2i\right|=\sqrt{4+4}=2\sqrt{2}$$
$$\arg\left(2+2i\right)=\arctan 1=\tfrac{\pi}{4}$$

b) $$\frac{\left(-2+2\sqrt{3}i\right)^4\left(-4-4i\right)^2}{1-i}$$

$$\frac{\left(-2+2\sqrt{3}i\right)^4\left(-4-4i\right)^2}{1-i}=\frac{\left[4\left(\cos\tfrac{2\pi}{3}+i\sin\tfrac{2\pi}{3}\right)\right]^4\left[4\sqrt{2}\left(\cos\tfrac{5\pi}{4}+i\sin\tfrac{5\pi}{4}\right)\right]^2}{\sqrt{2}\left(\cos\tfrac{-\pi}{4}+i\sin\tfrac{-\pi}{4}\right)}$$
$$=\frac{256\left(\cos\tfrac{8\pi}{3}+i\sin\tfrac{8\pi}{3}\right)32\left(\cos\tfrac{5\pi}{2}+i\sin\tfrac{5\pi}{2}\right)}{\sqrt{2}\left(\cos\tfrac{-\pi}{4}+i\sin\tfrac{-\pi}{4}\right)}$$
$$=4096\sqrt{2}\left(\cos\left(\tfrac{8\pi}{3}+\tfrac{5\pi}{2}+\tfrac{\pi}{4}\right)+i\sin\left(\tfrac{8\pi}{3}+\tfrac{5\pi}{2}+\tfrac{\pi}{4}\right)\right)$$
$$=4096\sqrt{2}\left(\cos\tfrac{-7\pi}{12}+i\sin\tfrac{-7\pi}{12}\right)$$
$$=4096\sqrt{2}e^{\frac{-7\pi}{12}i}$$

20. Find all 8[th] roots of $-i$ and plot them in the complex plane.

$$-i=\cos\tfrac{-\pi}{2}+i\sin\tfrac{-\pi}{2}$$
$$\left(-i\right)^{\frac{1}{8}}=\cos\frac{\frac{-\pi}{2}+2k\pi}{8}+i\sin\frac{\frac{-\pi}{2}+2k\pi}{8}$$
$$=\cos\tfrac{(4k-1)\pi}{16}+i\sin\tfrac{(4k-1)\pi}{16}\qquad k=0,1,2,3,4,5,6,7$$



21. Solve the equation $e^z = -\sqrt{3}+i$. Give all solutions.

$$e^z = -\sqrt{3}+i \qquad \text{Thus} \quad e^x = 2 \qquad y = \tfrac{5\pi}{6} + 2\pi k \quad k \in \mathbb{Z}$$

$$e^x e^{yi} = 2\left(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\right) \qquad x = \ln 2$$

Ergo, $z = \ln 2 + \left(\tfrac{5\pi}{6} + 2\pi k\right)i \quad k \in \mathbb{Z}$

22. Prove that $\arccos z = -i\ln\left(z+\sqrt{z^2-1}\right)$.

$$\arccos z = w$$

$$z = \cos w$$

$$z = \frac{e^{iw}+e^{-iw}}{2}$$

$$2z = k + \tfrac{1}{k} \qquad k = e^{iw}$$

$$k^2 - 2zk + 1 = 0$$

$$k = \frac{2z \pm \sqrt{4z^2-4}}{2}$$

$$k = z + \sqrt{z^2-1} \qquad \text{Taking the positive branch}$$

$$e^{iw} = z + \sqrt{z^2-1}$$

$$iw = \ln\left(z+\sqrt{z^2-1}\right)$$

$$w = -i\ln\left(z+\sqrt{z^2-1}\right)$$

$$\arccos z = -i\ln\left(z+\sqrt{z^2-1}\right)$$

23. Prove that $\operatorname{arcsec} z = -i\ln\left(\frac{1}{z}+\sqrt{\frac{1}{z^2}-1}\right)$. Use this result to evaluate $\operatorname{arcsec}(i)$.

$$\operatorname{arcsec} z = w$$

$$z = \sec w = \frac{1}{\cos w}$$

$$\cos w = \frac{1}{z}$$

$$w = \arccos\frac{1}{z}$$

$$\operatorname{arcsec} z = -i\ln\left(\frac{1}{z}+\sqrt{\frac{1}{z^2}-1}\right)$$

$$\operatorname{arc\,sec} i = -i\ln\left(\frac{1}{i}+\sqrt{\frac{1}{i^2}-1}\right) = -i\ln\left(-i+\sqrt{-2}\right) = -i\ln\left(\left(\sqrt{2}-1\right)i\right)$$
$$= -i\left(\ln\left(\sqrt{2}-1\right)+\tfrac{\pi}{2}i\right) = \tfrac{\pi}{2} - i\ln\left(\sqrt{2}-1\right)$$

24. Describe the function $f(z)=(1-i)(z+2-3i)$ as a composition of simple planar transformations. Then give the image of the circle $C=\{z \mid |z-1|=1\}$.

$$f(z)=(1-i)(z+2-3i) \qquad |1-i|=\sqrt{2} \qquad \arg(1-i)=\tfrac{-\pi}{4}$$
$$=(1-i)z-1-5i$$

Thus $f=T_{-1-5i}\circ D_{\sqrt{2}}\circ R_{\frac{-\pi}{4}}$

Since $f(1)=(1-i)-1-5i=-6i$ then $f(C)=\left\{z \mid |z-6i|=\sqrt{2}\right\}$

25. Find $\langle z_n\rangle$ in closed form and then find z_9.

a) $z_0=3-4i \qquad z_{n+1}=(-3+3i)z_n+5-10i$

This is an affine system. Fixed point: $Z=(-3+3i)Z+5-10i$

$$Z=\frac{5-10i}{4-3i}\cdot\frac{4+3i}{4+3i}=\frac{50-25i}{25}=2-i$$

Thus $z_n=k(-3+3i)^n+2-i \qquad z_0=3-4i=k+2-i$

$k=1-3i$

Ergo, $\langle z_n\rangle=\left\langle(1-3i)(-3+3i)^n+2-i\right\rangle_{n=0}^{\infty}$

For z_9 we have $z_9=(1-3i)(-3+3i)^9+2-i$

$$=(1-3i)\left(3\sqrt{2}\left(\cos\tfrac{3\pi}{4}+i\sin\tfrac{3\pi}{4}\right)\right)^9+2-i$$
$$=(1-3i)\left(314928\sqrt{2}\left(\cos\tfrac{27\pi}{4}+i\sin\tfrac{27\pi}{4}\right)\right)+2-i$$
$$=(1-3i)(-314928+314928i)+2-i$$
$$=629858+1259711i$$

b) $z_0 = 3-2i$ $\qquad z_{n+1} = T_{2\sqrt{3}+3i} \circ D_4 \circ R_{120^\circ}(z_n)$

$$z_{n+1} = T_{2\sqrt{3}+3i} \circ D_4 \circ R_{120^\circ}(z_n)$$
$$= 4(\cos 120^\circ + i\sin 120^\circ) z_n + 2\sqrt{3} + 3i$$
$$= \left(-2+2\sqrt{3}i\right) z_n + 2\sqrt{3} + 3i$$

This is an affine system.

Fixed point: $Z = \left(-2+2\sqrt{3}i\right) Z + 2\sqrt{3} + 3i$

$$Z = \frac{2\sqrt{3}+3i}{3-2\sqrt{3}i} \cdot \frac{3+2\sqrt{3}i}{3+2\sqrt{3}i} = \frac{21i}{21} = i$$

Thus $z_n = k\left(-2+2\sqrt{3}i\right)^n + i$ $\qquad z_0 = 3-2i = k+i$

$k = 3-3i$

Ergo, $\langle z_n \rangle = \left\langle (3-3i)\left(-2+2\sqrt{3}i\right)^n + i \right\rangle_{n=0}^{\infty}$

For z_9 we have $z_9 = (3-3i)\left(-2+2\sqrt{3}i\right)^9 + i$

$$= (3-3i)\left(4\left(\cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2}\right)\right)^9 + i$$
$$= (3-3i)\left(262144\left(\cos\tfrac{27\pi}{2} + i\sin\tfrac{27\pi}{2}\right)\right) + i$$
$$= (3-3i)(-262144i) + i$$
$$= 786432 - 786431i$$

26. Find polynomial q and r so that $p = qd + r$ and $\deg(r) < \deg(d)$.

a) $p(x) = 2x^5 - 6x^4 + 6x^3 + 14x^2 - 19x + 9$ $\qquad d(x) = x^2 - 3x + 5$

$$\begin{array}{r|l}
 & 2x^3 - 4x + 2 \\
\hline
x^2 - 3x + 5 & 2x^5 - 6x^4 + 6x^3 + 14x^2 - 19x + 9 \\
 & \underline{2x^5 - 6x^4 + 10x^3} \\
 & \qquad -4x^3 + 14x^2 - 19x + 9 \\
 & \qquad \underline{-4x^3 + 12x^2 - 20x} \\
 & \qquad\qquad 2x^2 + x + 9 \\
 & \qquad\qquad \underline{2x^2 - 6x + 10} \\
 & \qquad\qquad\qquad 7x - 1
\end{array}$$

Thus $q(x) = 2x^3 - 4x + 2$ and $r(x) = 7x - 1$

b) $p(z)=(7-i)z^5+(13+i)z^4+(8+6i)z^3+(-3-13i)z^2-8z+4+19i$

$d(z)=(3+i)z^2-2$

$$\begin{array}{r|l} & (2-i)z^3+(4-i)z^2+4z-5i \\ \hline (3+i)z^2-2 & (7-i)z^5+(13+i)z^4+(8+6i)z^3+(-3-13i)z^2-8z+4+19i \\ & \underline{(7-i)z^5 \qquad +(-4+2i)z^3} \\ & (13+i)z^4+(12+4i)z^3+(-3-13i)z^2-8z+4+19i \\ & \underline{(13+i)z^4 \qquad +(-8+2i)z^2} \\ & (12+4i)z^3+(5-15i)z^2-8z+4+19i \\ & \underline{(12+4i)z^3 \qquad -8z} \\ & (5-15i)z^2 \qquad +4+19i \\ & \underline{(5-15i)z^2 \qquad +10i} \\ & 4+9i \end{array}$$

Thus $q(z)=(2-i)z^3+(4-i)z^2+4z-5i$ and $r(z)=4+9i$

c) $p(x)=3x^4+\left(3-6\sqrt{2}\right)x^3-2\sqrt{2}x^2+\left(9-2\sqrt{2}\right)x+\sqrt{2}+1$ $\qquad d(x)=x-2\sqrt{2}+1$

$$\begin{array}{r|rrrrr} & 3 & 3-6\sqrt{2} & -2\sqrt{2} & 9-2\sqrt{2} & \sqrt{2}+1 \\ & & -3+6\sqrt{2} & 0 & -8+2\sqrt{2} & 2\sqrt{2}-1 \\ \hline 2\sqrt{2}-1 & 3 & 0 & -2\sqrt{2} & 1 & 3\sqrt{2} \end{array}$$

Thus $q(x)=3x^3-2\sqrt{2}x+1$ and $r(x)=3\sqrt{2}$

27. Factor *f* completely over (i) $\mathbb{Q}$ (ii) $\mathbb{R}$ and (iii) $\mathbb{C}$. (Give the prime factorization)

a) $f(x)=x^4-4x^3+3x^2+14x+26$ $\qquad 3-2i$ is a zero of *f*

$$\begin{array}{r|rrrrr} & 1 & -4 & 3 & 14 & 26 \\ & & 3-2i & -7-4i & -20-4i & -26 \\ \hline 3-2i & 1 & -1-2i & -4-4i & -6-4i & 0 \\ & & 3+2i & 6+4i & 6+4i & \\ \hline 3+2i & 1 & 2 & 2 & 0 & \end{array}$$

Thus $f(x)=(x-3+2i)(x-3-2i)\left(x^2+2x+2\right)$ $\qquad x^2+2x+2=0$

$$x=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$$

(i) and (ii) $f(x)=\left(x^2-6x+13\right)\left(x^2+2x+2\right)$

(iii) $f(x)=(x-3+2i)(x-3-2i)(x+1-i)(x+1+i)$

b) $f(z)=z^3+(6-4i)z^2+(-2-i)z+1+47i$ $\qquad z-2+i$ is a factor of f

$$\begin{array}{r|rrrr} & 1 & 6-4i & -2-i & 1+47i \\ & & 2-i & 11-18i & -1-47i \\ \hline 2-i & 1 & 8-5i & 9-19i & 0 \end{array}$$

$f(z)=(z-2+i)(z^2+(8-5i)z+9-19i)$

$z^2+(8-5i)z+9-19i=0$

$$z=\frac{-8+5i\pm\sqrt{(8-5i)^2-4(9-19i)}}{2}$$

$$=\frac{-8+5i\pm\sqrt{3-4i}}{2}$$

$3-4i=25\left(\cos\left(\arctan\frac{-4}{3}\right)+i\cos\left(\arctan\frac{-4}{3}\right)\right)$

$(3-4i)^{\frac{1}{2}}=\sqrt{25}\left(\cos\left(\frac{\arctan\frac{-4}{3}+2\pi k}{2}\right)+i\cos\left(\frac{\arctan\frac{-4}{3}+2\pi k}{2}\right)\right)$ $\qquad k=0,1$

$\sqrt{3-4i}=2-i$

Thus $z=\dfrac{-8+5i\pm(2-i)}{2}$

$z=-3+2i,-5+3i$

Ergo, $f(z)=(z-2+i)(z+3-2i)(z+5-3i)$ $\qquad$ (only iii applies)

c) $f(x)=x^6-2x^5+10x^4-16x^3+32x^2-32x+32$ $\qquad 2i$ is a zero of multiplicity 2

$$\begin{array}{r|rrrrrrr} & 1 & -2 & 10 & -16 & 32 & -32 & 32 \\ & & 2i & -4-4i & 8+12i & -24-16i & 32+16i & -32 \\ \hline 2i & 1 & -2+2i & 6-4i & -8+12i & 8-16i & 16i & 0 \\ & & -2i & 4i & -12i & 16i & -16i & \\ \hline -2i & 1 & -2 & 6 & -8 & 8 & 0 & \\ & & 2i & -4-4i & 8+4i & -8 & & \\ \hline 2i & 1 & -2+2i & 2-4i & 4i & 0 & & \\ & & -2i & 4i & -4i & & & \\ \hline -2i & 1 & -2 & 2 & 0 & & & \end{array}$$

Thus $f(x)=(x-2i)^2(x+2i)^2(x^2-2x+2)$ $\qquad x^2-2x+2=0$

$$x=\frac{2\pm\sqrt{-4}}{2}=1\pm i$$

(i) and (ii) $f(x)=(x^2+4)^2(x^2-2x+2)$

(iii) $f(x)=(x-2i)^2(x+2i)^2(x-1+i)(x-1-i)$

d) $f(x) = x^4 - \frac{25}{6}x^3 + \frac{3}{2}x^2 + \frac{1}{2}x - \frac{1}{6}$

$= \frac{1}{6}\left(6x^4 - 25x^3 + 9x^2 + 3x - 1\right)$

If $\frac{c}{d}$ is a reduced rational zero for *f*, then $c \mid 1 \quad \Rightarrow c \pm 1$

$d \mid 6 \quad \Rightarrow d = \pm 1, \pm 2, \pm 3, \pm 6$

Thus $\frac{c}{d} = \pm\frac{1}{6}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1$

If $x = \frac{1}{6}$

$$\begin{array}{c|ccccc} & 6 & -25 & 9 & 3 & -1 \\ & & 1 & -4 & \frac{5}{6} & \frac{23}{36} \\ \hline \frac{1}{6} & 6 & -24 & 5 & \frac{23}{6} & \frac{-13}{36} \end{array}$$

$f\left(\frac{1}{6}\right) \neq 0$

If $x = \frac{1}{3}$

$$\begin{array}{c|ccccc} & 6 & -25 & 9 & 3 & -1 \\ & & 2 & \frac{-23}{2} & \frac{-5}{9} & \frac{22}{27} \\ \hline \frac{1}{3} & 6 & -23 & \frac{-5}{3} & \frac{22}{9} & \frac{-5}{7} \end{array}$$

$f\left(\frac{1}{3}\right) \neq 0$

If $x = \frac{1}{2}$

$$\begin{array}{c|ccccc} & 6 & -25 & 9 & 3 & -1 \\ & & 3 & -11 & -1 & 1 \\ \hline \frac{1}{2} & 6 & -22 & -2 & 2 & 0 \end{array}$$

$f\left(\frac{1}{2}\right) = 0$

Thus $f(x) = \frac{1}{6}\left(x - \frac{1}{2}\right)\left(6x^3 - 22x^2 - 2x + 2\right)$

$= \frac{1}{3}\left(x - \frac{1}{2}\right)\left(3x^3 - 11x^2 - x + 1\right)$

If $\frac{c}{d}$ is a reduced rational zero for *f*, then $\frac{c}{d} = \pm\frac{1}{3}, \pm 1$

If $x = 1$

$$\begin{array}{c|cccc} & 3 & -11 & -1 & 1 \\ & & 3 & -8 & -9 \\ \hline 1 & 3 & -8 & -9 & -8 \end{array}$$

$f(1) \neq 0$

If $x = \frac{-1}{3}$

$$\begin{array}{c|cccc} & 3 & -11 & -1 & 1 \\ & & -1 & 4 & -1 \\ \hline \frac{-1}{3} & 3 & -12 & 3 & 0 \end{array}$$

$f\left(\frac{-1}{3}\right) = 0$

Thus $f(x) = \frac{1}{3}\left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right)\left(3x^2 - 12x + 3\right)$

$= \left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right)\left(x^2 - 4x + 1\right)$

$x^2 - 4x + 1 = 0$

$x = \frac{4 \pm \sqrt{16-4}}{2}$

$= 2 \pm \sqrt{3}$

(i) $f(x) = \left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right)\left(x^2 - 4x + 1\right)$

(ii) and (iii) $f(x) = \left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right)\left(x - 2 - \sqrt{3}\right)\left(x - 2 + \sqrt{3}\right)$

e) $f\left(x\right)=x^5-4x^4+x^3+10x^2-4x-8$

If $\frac{c}{d}$ is a reduced rational zero for f, then $c\,|\,8 \quad \Rightarrow c\pm1,\pm2,\pm4,\pm8$

$d\,|\,1 \quad \Rightarrow d=\pm1$

Thus $\frac{c}{d}=\pm1,\pm2,\pm4,\pm8$

If $x=1$

$$\begin{array}{r|rrrrrr} & 1 & -4 & 1 & 10 & -4 & -8 \\ & & 1 & -3 & -2 & 8 & 4 \\ \hline 1 & 1 & -3 & -2 & 8 & 4 & -4 \end{array}$$

$f\left(1\right)\neq 0$

If $x=2$

$$\begin{array}{r|rrrrrr} & 1 & -4 & 1 & 10 & -4 & -8 \\ & & 2 & -4 & -6 & 8 & 8 \\ \hline 2 & 1 & -2 & -3 & 4 & 4 & 0 \\ & & 2 & 0 & -6 & -4 & \\ \hline 2 & 1 & 0 & -3 & -2 & 0 & \\ & & 2 & 4 & 2 & & \\ \hline 2 & 1 & 2 & 1 & 0 & & \end{array}$$

Thus $f\left(x\right)=\left(x-2\right)^3\left(x^2+2x+1\right)$

(i) (ii) (iii) $f\left(x\right)=\left(x-2\right)^3\left(x+1\right)^2$

(ii)

f) $f\left(x\right)=5x^4+\frac{13}{2}x^3-29x^2-\frac{19}{2}x+3$

$=\frac{1}{2}\left(10x^4+13x^3-58x^2-19x+6\right)$

If $\frac{c}{d}$ is a reduced rational zero for f, then $c\,|\,6 \quad \Rightarrow c=\pm1,\pm2,\pm3,\pm6$

$d\,|\,10 \quad \Rightarrow d=\pm1,\pm2,\pm5,\pm10$

Thus $\frac{c}{d}=\pm\frac{1}{10},\pm\frac{1}{5},\pm\frac{3}{10},\pm\frac{2}{5},\pm\frac{1}{2},\pm\frac{3}{5},\pm1,\pm\frac{6}{5},\pm\frac{3}{2},\pm2,\pm3,\pm6$

If $x=\frac{1}{10}$

$$\begin{array}{r|rrrrr} & 10 & 13 & -58 & -19 & 6 \\ & & 1 & \frac{7}{5} & \frac{-283}{50} & \frac{-1233}{500} \\ \hline \frac{1}{10} & 10 & 14 & \frac{-283}{5} & \frac{-1233}{50} & \frac{1767}{500} \end{array}$$

$f\left(\frac{1}{10}\right)\neq 0$

If $x=\frac{1}{5}$

$$\begin{array}{r|rrrrr} & 10 & 13 & -58 & -19 & 6 \\ & & 2 & 3 & -11 & -6 \\ \hline \frac{1}{5} & 10 & 15 & -55 & -30 & 0 \end{array}$$

Thus $f\left(x\right)=\frac{1}{2}\left(x-\frac{1}{5}\right)\left(10x^3+15x^2-55x-30\right)$

$=\frac{5}{2}\left(x-\frac{1}{5}\right)\left(2x^3+3x^2-11x-6\right)$

If $\frac{c}{d}$ is a reduced rational zero for f, then $c\,|\,6 \quad \Rightarrow c=\pm1,\pm2,\pm3,\pm6$

$d\,|\,2 \quad \Rightarrow d=\pm1,\pm2$

Thus $\frac{c}{d}=\pm\frac{1}{2},\pm1,\pm\frac{3}{2},\pm2,\pm3,\pm6$

If $x=\frac{1}{2}$

$$\begin{array}{r|rrrr} & 2 & 3 & -11 & -6 \\ & & 1 & 2 & \frac{-9}{2} \\ \hline \frac{1}{2} & 2 & 4 & -9 & \frac{-21}{2} \end{array}$$

If $x=1$

$$\begin{array}{r|rrrr} & 2 & 3 & -11 & -6 \\ & & 2 & 5 & -6 \\ \hline 1 & 2 & 5 & -6 & -12 \end{array}$$

If $x=\frac{3}{2}$

$$\begin{array}{r|rrrr} & 2 & 3 & -11 & -6 \\ & & 3 & 9 & -3 \\ \hline \frac{3}{2} & 2 & 6 & -2 & -9 \end{array}$$

If $x=2$

$$\begin{array}{r|rrrr} & 2 & 3 & -11 & -6 \\ & & 4 & 14 & 6 \\ \hline 2 & 2 & 7 & 3 & 0 \end{array}$$

Thus $f(x)=\frac{5}{2}\left(x-\frac{1}{5}\right)(x-2)\left(2x^2+7x+3\right)$

$$2x^2+7x+3=0$$
$$x=\frac{-7\pm\sqrt{49-24}}{4}$$
$$x=-3,\tfrac{-1}{2}$$

(i) (ii) (iii) $f(x)=5(x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{5}\right)(x+3)$

g) $f(x)=x^6+3x^5-6x^3-19x^2-45x-30$

If $\frac{c}{d}$ is a reduced rational zero for *f*, then

$c\,|\,30 \quad \Rightarrow c=\pm1,\pm2,\pm3,\pm5,\pm6,\pm10,\pm15,\pm30$

$d\,|\,1 \quad \Rightarrow d=\pm1$

Thus $\frac{c}{d}=\pm1,\pm2,\pm3,\pm5,\pm6,\pm10,\pm15,\pm30$

If $x=1$

$$\begin{array}{r|rrrrrrr} & 1 & 3 & 0 & -6 & -19 & -45 & -30 \\ & & 1 & 4 & 4 & -2 & -21 & -66 \\ \hline 1 & 1 & 4 & 4 & -2 & -21 & -66 & -96 \end{array}$$

If $x=2$

$$\begin{array}{r|rrrrrrr} & 1 & 3 & 0 & -6 & -19 & -45 & -30 \\ & & 2 & 10 & 50 & 88 & 138 & 186 \\ \hline 2 & 1 & 5 & 10 & 44 & 69 & 93 & 156 \end{array}$$

Since all coefficients are ≥ 0, then there are no real zeros >2

If $x=-1$

$$\begin{array}{r|rrrrrrr} & 1 & 3 & 0 & -6 & -19 & -45 & -30 \\ & & -1 & -2 & 2 & 4 & 15 & 30 \\ \hline -1 & 1 & 2 & -2 & -4 & -15 & -30 & 0 \\ & & -1 & -1 & 3 & 1 & 14 & \\ \hline -1 & 1 & 1 & -3 & -1 & -14 & -16 & \end{array}$$

If $x=-2$

$$\begin{array}{r|rrrrrr} & 1 & 2 & -2 & -4 & -15 & -30 \\ & & -2 & 0 & 4 & 0 & 30 \\ \hline -2 & 1 & 0 & -2 & 0 & -15 & 0 \end{array}$$

Thus $f(x)=(x+1)(x+2)(x^4-2x^2-15)$

(i) $f(x)=(x+1)(x+2)(x^2+3)(x^2-5)$

(ii) $f(x)=(x+1)(x+2)(x^2+3)(x-\sqrt{5})(x+\sqrt{5})$

(iii) $f(x)=(x+1)(x+2)(x+\sqrt{3}i)(x-\sqrt{3}i)(x-\sqrt{5})(x+\sqrt{5})$

28. Find a zero to two-figure accuracy for x^3-7 using the bisection method.

$f(x)=x^3-7$ $\quad f(1)=-6<0$

$f(2)=1>0$

$x_1=1.5 \quad f(1.5)=-3.625<0 \quad \therefore c\in(1.5,2)$

$x_2=1.75 \quad f(1.75)=-1.64<0 \quad \therefore c\in(1.75,2)$

$x_3=1.875 \quad f(1.875)=-0.41<0 \quad \therefore c\in(1.875,2)$

$x_4=1.9375 \quad f(1.9375)=0.27>0 \quad \therefore c\in(1.875,1.9375)$

Ergo, $x\approx 1.9$

29. Find a zero to six-figure accuracy for x^3+4x^2-2, then use this zero to find all the other zeros (to five-figure accuracy).

$f(x)=x^3+4x^2-2$ $\quad f(0)=-2<0$

$f(1)=3>0$

Thus $x_0=0.5$ and $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}=x_n-\dfrac{x_n^3+4x_n^2-2}{3x_n^2+8x_n}=\dfrac{2x_n^3+4x_n^2+2}{3x_n^2+8x_n}$

$x_1=0.6842105263$

$x_2=0.6561672001$

$x_3=0.6554458608$

$x_4=0.6554423816$

$x_5=0.6554423816$

Ergo, $x\approx 0.655442$

$$\begin{array}{r|rrrr} & 1 & 4 & 0 & -2 \\ & & 0.655442 & 3.0513722 & 1.9999975 \\ \hline 0.655442 & 1 & 4.655442 & 3.0513722 & 0 \end{array}$$

Thus $f(x)=(x-0.655442)(x^2+4.65542x+3.0513722)$

$$x^2 + 4.65542x + 3.0513722 = 0$$
$$x = \frac{-4.65542 \pm \sqrt{4.65542 - 4(3.0513722)}}{2}$$
$$x \approx -3.8662, -0.78924$$

Hence the zeros of $x^3 + 4x^2 - 2$ are -3.8662, -0.78924, 0.65544.

30. A committee with four members is to be chosen among 7 women and 10 men. What is the probability that the committee will have more men than women?

$$P(\text{more men}) = P(3 \text{ men} + 1 \text{ women}) + P(4 \text{ men})$$
$$= \frac{\binom{10}{3}\binom{7}{1}}{\binom{17}{4}} + \frac{\binom{10}{4}}{\binom{17}{4}} = \frac{15}{34}$$

31. Two urns, A and B, have green and blue marbles. Urn A has 2 green marbles and 4 blue marbles, while urn B has 4 green marbles and 1 blue marble. A die is rolled. If a 1 or 2 comes up, 2 marbles are chosen from urn A, and if a 3, 4, 5 or 6 is chosen, 2 marbles are chosen from urn B.

a) What is the probability of having two green marbles?

$$P(G_1G_2) = P(AG_1G_2) + P(BG_1G_2)$$
$$= P(A)P(G_1 \mid A)P(G_2 \mid AG_1) + P(B)P(G_1 \mid B)P(G_2 \mid BG_1)$$
$$= \tfrac{1}{3}\tfrac{2}{6}\tfrac{1}{5} + \tfrac{2}{3}\tfrac{4}{5}\tfrac{3}{4} = \tfrac{19}{45}$$

b) What is the probability of having at least one green marble?

$$P(\text{at least one green}) = 1 - P(B_1B_2)$$
$$= 1 - \left(P(AB_1B_2) + P(BB_1B_2)\right)$$
$$= 1 - P(A)P(B_1 \mid A)P(B_2 \mid AB_1) - P(B)P(B_1 \mid B)P(B_2 \mid BB_1)$$
$$= 1 - \tfrac{1}{3}\tfrac{4}{6}\tfrac{3}{5} - \tfrac{2}{3}\tfrac{1}{5}0 = \tfrac{13}{15}$$

c) If two green marbles were chosen, what is the probability they came from urn A?

$$P(A \mid G_1G_2) = \frac{P(AG_1G_2)}{P(G_1G_2)} = \frac{\tfrac{1}{3}\tfrac{2}{6}\tfrac{1}{5}}{\tfrac{19}{45}} = \frac{1}{19}$$

32. Find the coefficient of x^4 in the expression $\left(x-\frac{2}{\sqrt{x}}\right)^{10}$

$$\left(x-\tfrac{2}{\sqrt{x}}\right)^{10}=\sum_{i=1}^{10}\binom{10}{i}x^{10-i}\left(\tfrac{-2}{\sqrt{x}}\right)^{i}=\sum_{i=1}^{n}\binom{10}{i}x^{10-\frac{3}{2}i}(-1)^{i}2^{i}$$

For the x^4 term, $10-\frac{3}{2}i=4$

$$i=4$$

Hence the coefficient is $\binom{10}{4}(-1)^4 2^4=3360$

33. How many five digit numbers are there (a number cannot start with zero),
   a) if there are no repetitions
      $9P(9,4)=27216$
   b) if the numbers start with a 2 and end with a 4 (without repetitions)
      $P(8,3)=336$
   c) if the numbers don't have any fives (without repetitions)
      $8P(8,4)=13440$
   d) if the number is divisible by 5 (without repetitions)
      (ends with 5) + (ends with 0) = $8P(8,3)+P(9,4)=5712$
   e) if the first digit is even and the last is odd?
      $4\cdot 5P(8,3)=6720$

34. In how many ways can 14 men be placed into 6 team where the first team has 3 members, the second team has 2 members, the third team has 3 members, and the fourth, fifth, and sixth teams each have 2 members?
   Members of a team are indistinguishable.
   Problem identical to #words using the letters AAABBCCCDDEEFF
   $\frac{14!}{3!2!3!2!2!2!}=151351200$

35. An insurance company estimates that people fall into two categories, the ones that are inclined to have accidents (*I*) and the ones that aren't (*N*). Statistics show that an individual in *I* has a probability of 0.4 to have an accident in the space of a year, and a probability of 0.2 for those who are in *N*. If 30% of the population is in *I*, what is the probability that
   a) A newly insured person has an accident during the next year.
$$\begin{aligned}P(A)&=P(A\cap I)+P(A\cap N)\\&=P(I)P(A\mid I)+P(N)P(A\mid N)\\&=0.30\cdot 0.4+0.70\cdot 0.2=0.26\end{aligned}$$
   b) A newly insured person is in the category *I* if he had an accident during the first year.
$$P(I\mid A)=\frac{P(A\cap I)}{P(A)}=\frac{0.30\cdot 0.4}{0.26}=0.46$$

36. Teams A and B are in the finals. (The first team that wins two games wins the tournament. No ties allowed). If the probability that team A wins a game is 40% (according to experts), what is the probability that

a) team A wins the tournament

$$P(A_1A_2)+P(A_1B_2A_3)+P(B_1A_2A_3)=0.4\cdot 0.4+0.4\cdot 0.6\cdot 0.4+0.6\cdot 0.4\cdot 0.4=0.352$$

b) team B wins at least one game

$$\begin{aligned}P(\text{B wins at least one game})&=1-P(\text{B loses all games})\\&=1-P(AA)\\&=1-0.4\cdot 0.4\\&=0.84\end{aligned}$$

c) team A wins the tournament and team B wins at least one game

$$P(A_1B_2A_3)+P(B_1A_2A_3)=0.4\cdot 0.6\cdot 0.4+0.6\cdot 0.4\cdot 0.4=0.192$$

d) team A wins the tournament knowing that team B will win at least one game.

$$\begin{aligned}P(\text{A wins}|\text{B wins at leas one game})&=\frac{P(\text{B wins one game and A two})}{P(\text{B wins at least one game})}\\&=\frac{0.192}{0.84}\\&=0.229\end{aligned}$$

37. At an iron foundry, it has been established that the sand used for molding iron casting is too wet 5% of the time and too dry 3% of the time. Also, defective castings occur 1% of the time when the sand has the correct amount of moisture, 7% of the time when the sand is too dry, and 30% of the time when the sand is too wet. Suppose a casting is selected at random and found to be defective, what is the probability the sand was too wet?

$$\begin{aligned}P(W\mid D)&=\frac{P(WD)}{P(D)}=\frac{P(W)P(D\mid W)}{P(W)P(D\mid W)+P(D)P(D\mid D)+P(N)P(D\mid N)}\\&=\frac{0.05\cdot 0.30}{0.05\cdot 0.30+0.03\cdot 0.07+0.92\cdot 0.01}\\&=0.5703\end{aligned}$$

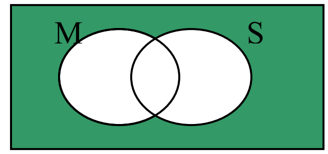
38. A group of 25 freshman mathematics students at a large university were surveyed. The results showed that 17 of the students read the *Mathematics Magazine*, 9 *Scientific American* and 5 both. A person from the group of freshman is selected at random,

a) Find the probability that the person reads the *Mathematics Magazine* or *Scientific American*.

$$\begin{aligned}P(M\cup S)&=P(M)+P(S)-P(M\cap S)\\&=\tfrac{17}{25}+\tfrac{9}{25}-\tfrac{5}{25}=\tfrac{21}{25}\end{aligned}$$

b) Find the probability that the person reads neither of these magazines.

$$P\left(\bar{M}\cap\bar{S}\right)=P\left(\overline{M\cup S}\right)$$
$$=1-P\left(M\cup S\right)$$
$$=1-\tfrac{21}{25}$$
$$=\tfrac{4}{25}$$



c) Find the probability that the person reads the *Mathematics Magazine* given that he reads *Scientific American*.

$$P\left(M\mid S\right)=\frac{P\left(M\cap S\right)}{P\left(S\right)}=\frac{\frac{5}{25}}{\frac{9}{25}}=\frac{5}{9}$$

d) Find the probability that the person does not read the *Mathematics Magazine* given that he does not read *Scientific American*.

$$P\left(\bar{M}\mid\bar{S}\right)=\frac{P\left(\bar{M}\cap\bar{S}\right)}{P\left(\bar{S}\right)}=\frac{P\left(\bar{M}\cap\bar{S}\right)}{1-P\left(S\right)}=\frac{\frac{4}{25}}{1-\frac{9}{25}}=\frac{1}{4}$$

39. The probability that a student applying for engineering at university is accepted is 0.5, the probability of being accepted in law is 0.4 and the probability of being accepted in engineering, law or both is 0.6. What is the probability that a person will be accepted

a) in both?

$$P\left(E\cup L\right)=P\left(E\right)+P\left(L\right)-P\left(E\cap L\right)$$
$$0.6=0.5+0.4-P\left(E\cap L\right)$$
$$P\left(E\cap L\right)=0.3$$

b) in law but not in engineering?

$$P\left(L\right)=P\left(L\cap E\right)+P\left(L\cap\bar{E}\right)$$
$$P\left(L\cap\bar{E}\right)=P\left(L\right)-P\left(L\cap E\right)$$
$$=0.4-0.3=0.1$$

40. Studies have shown that if a person had a car accident one year, the probability they will have another one the next year is 0.5, while if they didn't have any one year, then the probability they will not have one the next is 0.9. If Paul had an accident this year,

a) what is the probability that he will have a car accident *n* years from now?

$$P\left(A_{n+1}\right)=P\left(A_n\cap A_{n+1}\right)+P\left(\bar{A}_n\cap A_{n+1}\right)$$
$$=P\left(A_n\right)P\left(A_{n+1}\mid A_n\right)+P\left(\bar{A}_n\right)P\left(A_{n+1}\mid\bar{A}_n\right)$$
$$=P\left(A_n\right)0.5+\left(1-P\left(A_n\right)\right)\left(1-0.9\right)$$
$$=0.4P\left(A_n\right)+0.1$$

This is an affine dynamical system

Fixed point: $A=0.4A+0.1$

$A=\tfrac{1}{6}$

Thus $P(A_n)=k\left(\frac{2}{5}\right)^n+\frac{1}{6}$

$P(A_0)=1=k+\frac{1}{6}$

$k=\frac{5}{6}$

Ergo, $P(A_n)=\frac{5}{6}\left(\frac{2}{5}\right)^n+\frac{1}{6}$

b) In the long run, what is the proportion of years for which Paul will have a car accident?

$P(A_n)\to\dfrac{1}{6}$, thus Paul will have an accident once every six years

41. The probability a student will be late one class is 0.3 if he was late the previous class, and 0.05 if he wasn't late the previous class. Suppose Leonard is late for class today.

a) What is the probability that he will be late for class *n* classes from now?

$$\begin{aligned}P(L_{n+1})&=P(L_n\cap L_{n+1})+P(\overline{L}_n\cap L_{n+1})\\&=P(L_n)P(L_{n+1}\mid L_n)+P(\overline{L}_n)P(L_{n+1}\mid \overline{L}_n)\\&=P(L_n)0.3+(1-P(L_n))0.05\\&=0.25P(L_n)+0.05\end{aligned}$$

This is an affine dynamical system

Fixed point: $L=0.25L+0.05$

$L=\frac{1}{15}$

Thus $P(L_n)=k\left(\frac{1}{4}\right)^n+\frac{1}{15}$

$P(L_0)=1=k+\frac{1}{15}$

$k=\frac{14}{15}$

Ergo, $P(L_n)=\frac{14}{15}\left(\frac{1}{4}\right)^n+\frac{1}{15}$

b) In the long run, what is the proportion of classes for which Leonard will be late?

$P(L_n)\to\dfrac{1}{15}$, thus Leonard will be late a class out of fifteen