MATHEMATICS 201-BNJ-05
Topics in Mathematics
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Winter 2009

Assignment #3

SOLUTIONS

This assignment is due ***Friday March 27, 2009*** at the beginning of the class.
Complete solutions with exact answers are expected.

For questions involving Maple, a print-out of your work is expected, where your name is written in the Worksheet, each question is clearly labeled, and the answers are clearly presented. Also, you must copy your file in my "TEST" subfolder (W:\Tests\mhuard\Topics\Assignment 3), where your name should be included in the name of the file (for example: Assignment 3 – Your Name).

Do the following questions from the book.

5.2.42 page 120 (2 points)

$$\frac{(1-i)(1+2i)(1-3i)}{(1+i)(1-2i)(1+3i)}=\frac{(3+i)(1-3i)}{(3-i)(1+3i)}=\frac{6-8i}{6+8i}\cdot\frac{6-8i}{6-8i}=\frac{-28-96i}{100}=\tfrac{-7}{25}-\tfrac{24}{25}i$$

5.2.56 page 120 (3 points)

Let $w=a+bi$ and $z=c+di$.

$$\left|\frac{w}{z}\right|=\left|\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}\right|=\left|\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\right|$$

$$=\frac{\sqrt{(ac+bd)^2+(bc-ad)^2}}{c^2+d^2}$$

$$=\frac{\sqrt{a^2c^2+2abcd+b^2d^2+b^2c^2-2abcd+a^2d^2}}{c^2+d^2}$$

$$=\frac{\sqrt{a^2c^2+b^2d^2+b^2c^2+a^2d^2}}{c^2+d^2}$$

$$=\frac{\sqrt{a^2(c^2+d^2)+b^2(d^2+c^2)}}{c^2+d^2}$$

$$=\frac{\sqrt{(a^2+b^2)(c^2+d^2)}}{c^2+d^2}$$

$$=\frac{\sqrt{(a^2+b^2)}}{\sqrt{(c^2+d^2)}}=\frac{|w|}{|z|}$$

5.3.42 page 128 (2 points)

$$|1+i|=\sqrt{2} \qquad \arg(1+i)=\tfrac{\pi}{4}$$

$$|1-i|=\sqrt{2} \qquad \arg(1-i)=-\tfrac{\pi}{4}$$

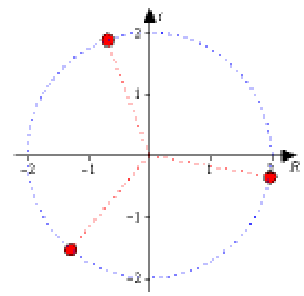
$$\left(\frac{1+i}{1-i}\right)^{15}=\left(\frac{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{\sqrt{2}\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right)}\right)^{15}$$

$$=\left(\cos\left(\tfrac{\pi}{4}-\left(\tfrac{-\pi}{4}\right)\right)+i\sin\left(\tfrac{\pi}{4}-\left(\tfrac{-\pi}{4}\right)\right)\right)^{15}$$

$$=\cos\tfrac{15\pi}{2}+i\sin\tfrac{15\pi}{2}$$

$$=\cos\tfrac{3\pi}{2}+i\sin\tfrac{3\pi}{2}$$

5.3.64 page 129 (2 points)



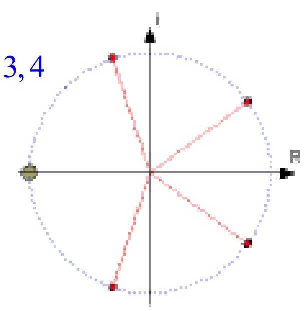
$$\left|4\sqrt{3}-4i\right|=8 \qquad \operatorname{Arg}\left(4\sqrt{3}-4i\right)=\tfrac{-\pi}{6}$$

$$4\sqrt{3}-4i=8\left(\cos\tfrac{-\pi}{6}+i\sin\tfrac{-\pi}{6}\right)$$

$$\left(4\sqrt{3}-4i\right)^{\frac{1}{3}}=\sqrt[3]{8}\left(\cos\frac{\frac{-\pi}{6}+2\pi k}{3}+i\sin\frac{\frac{-\pi}{6}+2\pi k}{3}\right) \qquad k=0,1,2$$

$$=2\left(\cos\left(\tfrac{-\pi}{18}+\tfrac{2\pi}{3}k\right)+i\sin\left(\tfrac{-\pi}{18}+\tfrac{2\pi}{3}k\right)\right) \quad k=0,1,2$$

5.4.30 page 135 (2 points)



$$\left(e^{5-\pi i}\right)^{\frac{1}{5}}=\left(e^{5}\left(\cos(-\pi)+i\sin(-\pi)\right)\right)^{\frac{1}{5}}$$

$$=e\left(\cos\tfrac{-\pi+2\pi k}{5}+i\sin\tfrac{-\pi+2\pi k}{5}\right) \qquad k=0,1,2,3,4$$

Since $\operatorname{Arg}\left(e^{5-\pi i}\right)=\pi$,

then, with $k=3$,

$$\sqrt[5]{e^{5-\pi i}}=e\left(\cos\pi+i\sin\pi\right)=-e$$

5.4.38 page 135 (4 points)

a) $z=re^{i\theta}=re^{i(\theta+2\pi k)}=e^{\ln r}e^{i(\theta+2\pi k)}=e^{\ln r+i(\theta+2\pi k)} \qquad k\in\mathbb{Z}$

$$z^{i}=\left(e^{\ln r+i(\theta+2\pi k)}\right)^{i}=e^{i(\ln r+i(\theta+2k\pi))}=e^{-(\theta+2k\pi)+i\ln r}$$

$$=e^{-(\theta+2k\pi)}\left(\cos(\ln r)+i\sin(\ln r)\right) \qquad k\in\mathbb{Z}$$

b) $i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$$i^{i}=e^{\frac{-\pi}{2}}\left(\cos(\ln 1)+i\sin(\ln 1)\right)$$

$$=e^{\frac{-\pi}{2}}$$

5.5.6 page 143 (4 points)

$$f(z) = (1+i)(z-3+4i)$$
$$= (1+i)z + (1+i)(-3+4i)$$
$$= (1+i)z - 7 + i$$

$\operatorname{Arg}(1+i) = \frac{\pi}{4}$ $\qquad |1+i| = \sqrt{2}$

Since f is an affine function, then $f = T_{-7+i} \circ D_{\sqrt{2}} \circ R_{\frac{\pi}{4}}$

Thus a $\frac{\pi}{4}$ rotation, followed by a dilation of factor $\sqrt{2}$, then a $-7+i$ translation.

Since $f(1) = -6+2i$ then

$f(C) = \left\{ z \mid |z+6-2i| = \sqrt{2} \right\}$, circle of radius $\sqrt{2}$ centered at $-6+2i$

5.5.38 page 145 + questions 44, 67, 68 (relating to 38) (12 points)

This is an affine system with

fixed point $Z = \left(\frac{3}{5}+\frac{3}{5}i\right)Z - 5i$

$$Z = \frac{-5i}{\frac{2}{5}-\frac{3}{5}i} \cdot \frac{\frac{2}{5}+\frac{3}{5}i}{\frac{2}{5}+\frac{3}{5}i} = \frac{3-2i}{\frac{13}{25}} = \frac{75}{13} - \frac{50}{13}i$$

Thus $\langle z_n \rangle = \left\langle k\left(\frac{3}{5}+\frac{3}{5}i\right)^n + \frac{75}{13} - \frac{50}{13}i \right\rangle_{n=0}^{\infty}$

$$z_0 = 1+2i = k + \frac{75}{13} - \frac{50}{13}i$$
$$k = \frac{-62}{13} + \frac{76}{13}i$$

Ergo $\langle z_n \rangle = \left\langle \left(\frac{-62}{13} + \frac{76}{13}\right)\left(\frac{3}{5}+\frac{3}{5}i\right)^n + \frac{75}{13} - \frac{50}{13}i \right\rangle_{n=0}^{\infty}$

5.5.44 $\arg\left(\frac{3}{5}+\frac{3}{5}i\right) = \frac{\pi}{4}$ $\qquad \left|\frac{3}{5}+\frac{3}{5}i\right| = \frac{3\sqrt{2}}{5}$

$$\left(\frac{3}{5}+\frac{3}{5}i\right)^{20} = \left(\frac{3\sqrt{2}}{5}\right)^{20}\left(\cos\frac{20\pi}{4} + i\sin\frac{20\pi}{4}\right) = -\left(\frac{3\sqrt{2}}{5}\right)^{20}$$

$$z_{20} = \left(\frac{-62}{13}+\frac{76}{13}i\right)\left(\frac{3}{5}+\frac{3}{5}i\right)^{20} + \frac{75}{13} - \frac{50}{13}i$$
$$= \left(\frac{-62}{13}+\frac{76}{13}i\right)\left(-\left(\frac{3\sqrt{2}}{5}\right)^{20}\right) + \frac{75}{13} - \frac{50}{13}i$$
$$\approx 5.9478 - 4.0650i$$

5.5.67
```
> rsolve({z(n+1)=(3/5+3*I/5)*z(n)-5*I,z(0)=1+2*I},z);
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$$\left(-\frac{62}{13}+\frac{76}{13}I\right)\left(\frac{3}{5}+\frac{3}{5}I\right)^n + \left(\frac{75}{13}-\frac{50}{13}I\right)$$

5.5.68
```
> a:=n->(-62/13+76/13*I)*(3/5+3/5*I)^n+(75/13-50/13*I);
```
$$a := n \to \left(-\frac{62}{13}+\frac{76}{13}I\right)\left(\frac{3}{5}+\frac{3}{5}I\right)^n+\left(\frac{75}{13}-\frac{50}{13}I\right)$$

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> a(20);
```
$$\frac{567225103161351}{95367431640625}-\frac{387671314711898}{95367431640625}I$$

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> evalf(%,5);
```
$$5.9478 - 4.0650\,I$$

6.1.34 page 160 (4 points)

$$\begin{array}{rl}
 & 2z^2+(-1-5i)z+(-10+7i) \\
(1+i)z^2+(-2+3i)z-2i \,\big)\!\! & (2+2i)z^4 \qquad\qquad -7i \\
 & \underline{(2+2i)z^4+(-4+6i)z^3-4iz^2} \\
 & (4-6i)z^3+4iz^2-7i \\
 & \underline{(4-6i)z^3+(17+7i)z^2+(-10+2i)z} \\
 & (-17-3i)z^2+(10-2i)z-7i \\
 & \underline{(-17-3i)z^2+(-1-44i)z+14+20i} \\
 & (11+42i)z-14-27i
\end{array}$$

Thus $q(z)=2z^2+(-1-5i)z+(-10+7i)$

and $r(z)=(11+42i)z-14-27i$

6.2.18 page 163 (3 points)

$$\begin{array}{r|rrrrrr}
 & 1 & -i & -8-4i & -4+2i & 1+3i & 0 \\
 & & 3+i & 9+3i & 4-2i & 0 & 10i \\
\hline
3+i & 1 & 3 & 1-i & 0 & 1+3i & 10i
\end{array}$$

Thus $q(z)=z^4+3z^3+(1-i)z^2+1+3i$

and $r(z)=10i$

6.2.32 page 164 (4 points)

$$\begin{array}{r|rrrrr} & 1 & i & -4 & 10-4i & 10i \\ & & i & -2 & -6i & 10+10i \\ \hline i & 1 & 2i & -6 & 10-10i & 10+20i \end{array}$$

Thus $p(i)=10+20i$

$$\begin{array}{r|rrrrr} & 1 & i & -4 & 10-4i & 10i \\ & & -i & 0 & 4i & -10i \\ \hline -i & 1 & 0 & -4 & 10 & 0 \end{array}$$

Thus $p(-i)=0$

$$\begin{array}{r|rrrrr} & 1 & i & -4 & 10-4i & 10i \\ & & 1-2i & -1-3i & -11+7i & 5+5i \\ \hline 1-2i & 1 & 1-i & -5-3i & -1+3i & 5+15i \end{array}$$

Thus $p(1-2i)=5+15i$

6.3.50 page 169 (4 points)

$$\begin{aligned} 8x^6+7x^3-1 &= (8x^3-1)(x^3+1) \\ &= (2x-1)(4x^2+2x+1)(x+1)(x^2-x+1) \end{aligned}$$

$4x^2+2x+1=0$

$$x=\frac{-2\pm\sqrt{4-4\cdot 4}}{2\cdot 4}=\tfrac{-1}{4}\pm\tfrac{\sqrt{3}}{2}i$$

$x^2-x+1=0$

$$x=\frac{1\pm\sqrt{1-4}}{2}=\tfrac{1}{2}\pm\tfrac{\sqrt{3}}{2}i$$

Over $\mathbb{Q}$ and $\mathbb{R}$: $8x^6+7x^3-1=8\left(x-\tfrac{1}{2}\right)\left(x^2+\tfrac{1}{2}x+\tfrac{1}{4}\right)(x+1)(x^2-x+1)$

Over $\mathbb{C}$:

$$\begin{aligned} 8x^6+7x^3-1 &= 8\left(x-\tfrac{1}{2}\right)\left(x-\left(\tfrac{-1}{4}-\tfrac{\sqrt{3}}{2}i\right)\right)\left(x-\left(\tfrac{-1}{4}+\tfrac{\sqrt{3}}{2}i\right)\right)(x+1)\left(x-\left(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\right)\left(x-\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right) \\ &= 8\left(x-\tfrac{1}{2}\right)\left(x+\tfrac{1}{4}+\tfrac{\sqrt{3}}{4}i\right)\left(x+\tfrac{1}{4}-\tfrac{\sqrt{3}}{4}i\right)(x+1)\left(x-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\left(x-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right) \end{aligned}$$

6.3.60 page 170 (4 points)

$p(x) = 3x^4 - 2x^3 + 2x - 3 \qquad p(1) = 0$

$$\begin{array}{r|rrrrr} & 3 & -2 & 0 & 2 & -3 \\ & & 3 & 1 & 1 & 3 \\ \hline 1 & 3 & 1 & 1 & 3 & 0 \end{array}$$

Thus $p(x) = (x-1)(3x^3 + x^2 + x + 3)$. $\qquad p(-1) = 0$

$$\begin{array}{r|rrrr} & 3 & 1 & 1 & 3 \\ & & -3 & 3 & -3 \\ \hline -1 & 3 & -2 & 3 & 0 \end{array}$$

Hence $p(x) = (x-1)(x+1)(3x^2 - 2x + 3)$

Solving $3x^2 - 2x + 3 = 0$

$$x = \frac{2 \pm \sqrt{4 - 4\cdot 3\cdot 3}}{2\cdot 3} = \frac{2 \pm 4\sqrt{2}i}{6} = \tfrac{1}{3} \pm \tfrac{2\sqrt{2}}{3}i$$

Thus, the complete factorization over $\mathbb{Q}$ and $\mathbb{R}$ is

$$3x^4 - 2x^3 + 2x - 3 = 3(x-1)(x+1)\left(x^2 - \tfrac{2}{3}x + 1\right)$$

And over $\mathbb{C}$ is

$$\begin{aligned} 3x^4 - 2x^3 + 2x - 3 &= 3(x-1)(x+1)\left(x - \left(\tfrac{1}{3} + \tfrac{2\sqrt{2}}{3}i\right)\right)\left(x - \left(\tfrac{1}{3} - \tfrac{2\sqrt{2}}{3}i\right)\right) \\ &= 3(x-1)(x+1)\left(x - \tfrac{1}{3} - \tfrac{2\sqrt{2}}{3}i\right)\left(x - \tfrac{1}{3} + \tfrac{2\sqrt{2}}{3}i\right) \end{aligned}$$