MATHEMATICS 201-BNJ-05
Topics in Mathematics
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Winter 2009

Assignment #2
SOLUTIONS

This assignment is due ***Tuesday February 24, 2009*** at the beginning of the class.
Complete solutions with exact answers are expected.

Part I

Do the following questions from the book.

2.3.16 page 7 (7 points)

Characteristic equation $x^2 = 6x - 5$

$$x^2 - 6x + 5 = 0$$
$$(x-5)(x-1) = 0$$
$$x = 1, 5$$

Fixed point: $C = 6C - 5C + 4$, no solution.
Hence we can suspect that

$$c_n = k\left(5^n\right) + c + dn \quad (1)$$

Using mathematical induction

(i) $n = 0$ $\quad LS(1) = c_0 = -2$

$$RS(1) = k + c = LS(1) \text{ if } k + c = -2$$

$n = 1$ $\quad LS(1) = c_1 = 1$

$$RS(1) = 5k + c + d = LS(1) \text{ if } 5k + c + d = 1$$

(ii) Suppose $c_m = k\left(5^m\right) + c + dm$ (2)

and $c_{m+1} = k\left(5^{m+1}\right) + c + dm + d$ (3)

We must show that $c_{m+2} = k\left(5^{m+2}\right) + c + dm + 2d$ (4)

$$LS(4) = c_{m+2}$$
$$= 6c_{m+1} - 5c_m + 4$$
$$= 6\left(k\left(5^{m+1}\right) + c + dm + d\right) - 5\left(k\left(5^m\right) + c + dm\right) + 4 \quad \text{by } (2),(3)$$
$$= 6k\left(5^{m+1}\right) + 6c + 6dm + 6d - k\left(5^{m+1}\right) - 5c - 5dm + 4$$
$$= k\left(5^{m+2}\right) + c + dm + 6d + 4$$
$$= RS(4) \quad \text{if} \quad 6d + 4 = 2d$$

Hence $d = -1$.

From step (i) we have $k+c=-2$, which gives us $k=1$

$5k+c=2$ $\qquad$ $c=-3$

Ergo, by AMI, $\langle c_n \rangle = \langle 5^m - n - 3 \rangle_{n=0}^{\infty}$

3.2.12 page 70 (6 points)

To prove : $\sum_{k=1}^{n}\sum_{i=1}^{k} i = \frac{n(n+1)(n+2)}{6}$ (1)

(i) $n=1$ $\qquad$ LS of (1) $= \sum_{k=1}^{1}\sum_{i=1}^{k} i = \sum_{i=1}^{1} i = 1$

RS of (1) $= \frac{1(1+1)(2+1)}{6} = 1 =$ LS of (1)

(ii) Suppose $\sum_{k=1}^{m}\sum_{i=1}^{k} i = \frac{m(m+1)(m+2)}{6}$ (2)

Show that $\sum_{k=1}^{m+1}\sum_{i=1}^{k} i = \frac{(m+1)(m+2)(m+3)}{6}$ (3)

$$\text{LS of (3)} = \sum_{k=1}^{m+1}\sum_{i=1}^{k} i$$

$$= \sum_{k=1}^{m}\sum_{i=1}^{k} i + \sum_{k=m+1}^{m+1}\sum_{i=1}^{k} i$$

$$= \sum_{k=1}^{m}\sum_{i=1}^{k} i + \sum_{i=1}^{m+1} i$$

$$= \frac{m(m+1)(m+2)}{6} + \frac{(m+1)(m+2)}{2} \quad \text{by (2) and Arithmetic Series}$$

$$= \frac{m(m+1)(m+2)+3(m+1)(m+2)}{6}$$

$$= \frac{(m+1)(m+2)(m+3)}{6}$$

$$= \text{RS of (3)}$$

Ergo, by AMI, $\sum_{k=1}^{n}\sum_{i=1}^{k} i = \frac{n(n+1)(n+2)}{6}$

3.2.32 page 72 (2 points)

$$6+2\left(\tfrac{2}{3}6\right)+2\left(\tfrac{2}{3}\tfrac{2}{3}6\right)+\ldots+2\left(\tfrac{2}{3}\right)^9 6 = 6+\sum_{i=0}^{8} 8\left(\tfrac{2}{3}\right)^i$$

$$= 6+\frac{8\left(1-\left(\tfrac{2}{3}\right)^9\right)}{1-\tfrac{2}{3}} \qquad \text{Geometric Series}$$

$$= 6+24\left(1-\left(\tfrac{2}{3}\right)^9\right)$$

$$\approx 29.376 \text{ meters}$$

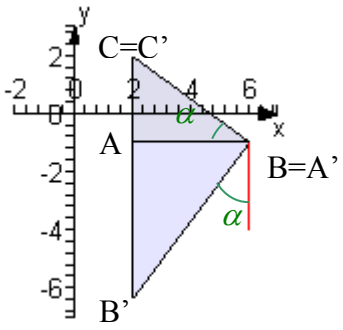
3.3.18 page 76 (2 points)

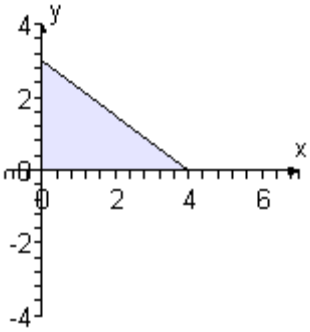
$$-1.1\overline{1001} = -1.1-\sum_{n=0}^{\infty}\tfrac{1001}{10^5}\left(10^{-4}\right)^n \qquad \text{geometric series with } |r| = \tfrac{1}{10^4} < 1$$

$$= -\frac{11}{10}-\frac{\frac{1001}{10^5}}{1-\frac{1}{10^4}} = -\frac{11}{10}-\frac{1001}{99990} = -\frac{1009}{909}$$

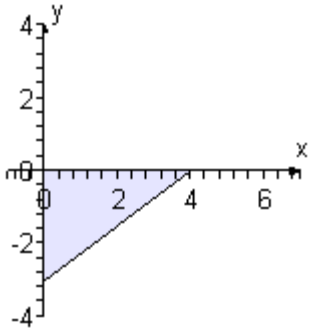
4.1.26 page 85 (7 points)

Start with a translation $T_{(-2,1)}$





Reflection across the x-axis : M_x

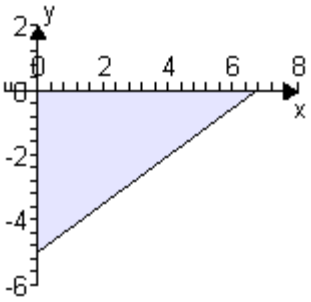


Dilation :

Since $|A'C'| = 5$

$|AC| = 3$

then $D_{\frac{5}{3}}$



Rotation $R_{-\beta}$, $\alpha = \arctan\frac{3}{4}$

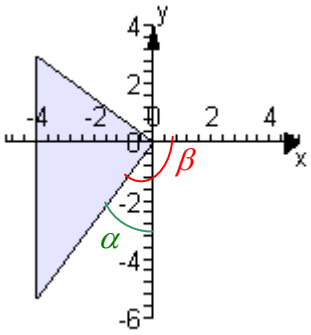
$\beta = \alpha + \frac{\pi}{2} = \arctan\frac{3}{4} + \frac{\pi}{2}$

$\cos(-\beta) = \cos\left(-\alpha - \frac{\pi}{2}\right)$

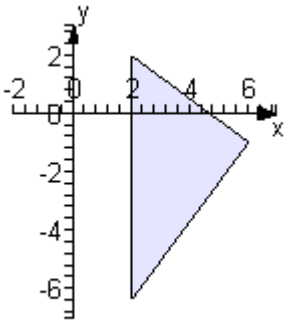
$= -\sin\alpha = \frac{-3}{5}$

$\sin(-\beta) = \sin\left(-\alpha - \frac{\pi}{2}\right)$

$= -\cos\alpha = \frac{-4}{5}$



Translation $T_{(6,-1)}$



$h(x,y) = T_{(6,-1)} \circ R_{-\beta} \circ D_{\frac{5}{3}} \circ M_x \circ T_{(-2,1)}(x,y)$

$= T_{(6,-1)} \circ R_{-\beta} \circ D_{\frac{5}{3}} \circ M_x (x-2, y+1)$

$= T_{(6,-1)} \circ R_{-\beta} \circ D_{\frac{5}{3}} (x-2, -y-1)$

$= T_{(6,-1)} \circ R_{-\beta} \left(\frac{5}{3}x - \frac{10}{3}, -\frac{5}{3}y - \frac{5}{3}\right)$

$= T_{(6,-1)} \left(\left(\frac{5}{3}x - \frac{10}{3}\right)\cos(-\beta) - \left(\frac{-5}{3}y - \frac{5}{3}\right)\sin(-\beta), \left(\frac{5}{3}x - \frac{10}{3}\right)\sin(-\beta) + \left(\frac{-5}{3}y - \frac{5}{3}\right)\cos(-\beta)\right)$

$= \left(\left(\frac{5}{3}x - \frac{10}{3}\right)\left(\frac{-3}{5}\right) - \left(\frac{-5}{3}y - \frac{5}{3}\right)\left(\frac{-4}{5}\right) + 6, \left(\frac{5}{3}x - \frac{10}{3}\right)\left(\frac{-4}{5}\right) + \left(\frac{-5}{3}y - \frac{5}{3}\right)\left(\frac{-3}{5}\right) - 1\right)$

$= \left(-x - \frac{4}{3}y + \frac{20}{3}, \frac{-4}{3}x + y + \frac{8}{3}\right)$

4.2.14 page 91 + question 20 (related to 14) (4 points)

The planar transformation has no inverse since $\det(F) = 0$

$$F\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 6u+4v \\ -3u-2v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$6u+4v = x$ $\qquad$ $x+2y=0$

$-3u-2v = y$ $\qquad$ $y = \frac{-1}{2}x$

Hence the image of the line $y = 2x+7$ is $y = \frac{-1}{2}x$

Is the image of $y = 2x+7$ *all* of $y = \frac{-1}{2}x$?

Choosing a point (x, y) on the line, we have

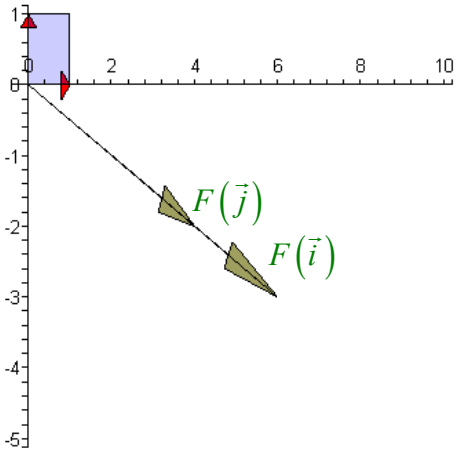
$$\begin{aligned} 6u+4v &= x \\ -3u-2v &= y \end{aligned} \qquad \text{and } v = 2u+7$$

Solving, we obtain $u = \frac{1}{14}x - 2$ and $v = \frac{1}{7}x + 3$.

Hence, the image of $y = 2x+7$ under F is $y = \frac{-1}{2}x$.

$$F\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$F\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$



4.3.26 page 96 + questions 34 and 35 (relating to 28) (8 points)

We have $x_{n+1} = -\frac{1}{2}x_n + y_n - 2$ (1)

$y_{n+1} = x_n - \frac{1}{2}y_n$ (2)

Solving in (1), $y_n = x_{n+1} + \frac{1}{2}x_n + 2$

$y_{n+1} = x_{n+2} + \frac{1}{2}x_{n+1} + 2$

Back in (2), $x_{n+2} + \frac{1}{2}x_{n+1} + 2 = x_n - \frac{1}{2}\left(x_{n+1} + \frac{1}{2}x_n + 2\right)$

$x_{n+2} = -x_{n+1} + \frac{3}{4}x_n - 3$

This is a second order affine dynamical system,

Characteristic equation: $x^2 = -x + \frac{3}{4}$

$4x^2 + 4x - 3 = 0$

$(2x+3)(2x-1) = 0$

$x = \frac{-3}{2}, \frac{1}{2}$

Fixed point: $X = -X + \frac{3}{4}X - 3$

$X = -\frac{12}{5}$

Ergo, $x_n = k\left(\frac{-3}{2}\right)^n + l\left(\frac{1}{2}\right)^n - \frac{12}{5}$

Also, $y_n = \left(k\left(\frac{-3}{2}\right)^{n+1} + l\left(\frac{1}{2}\right)^{n+1} - \frac{12}{5}\right) + \frac{1}{2}\left(k\left(\frac{-3}{2}\right)^n + l\left(\frac{1}{2}\right)^n - \frac{12}{5}\right) + 2$

$= \left(-k\left(\frac{-3}{2}\right)^n + l\left(\frac{1}{2}\right)^n - \frac{8}{5}\right)$

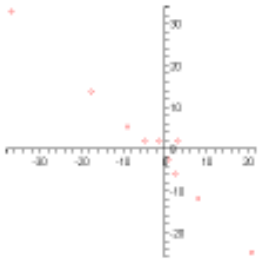
Since $(x_0, y_0) = (3, 2)$, we have

$3 = k + l - \frac{12}{5}$ $\quad k = \frac{9}{10}$

$2 = -k + l - \frac{8}{5}$ $\quad l = \frac{9}{2}$

Ergo, $\langle (x_n, y_n) \rangle = \left\langle \left(\frac{9}{10}\left(\frac{-3}{2}\right)^n + \frac{9}{2}\left(\frac{1}{2}\right)^n - \frac{12}{5}, \frac{-9}{10}\left(\frac{-3}{2}\right)^n + \frac{9}{2}\left(\frac{1}{2}\right)^n - \frac{8}{5}\right) \right\rangle_{n=0}^{\infty}$

4.3.34

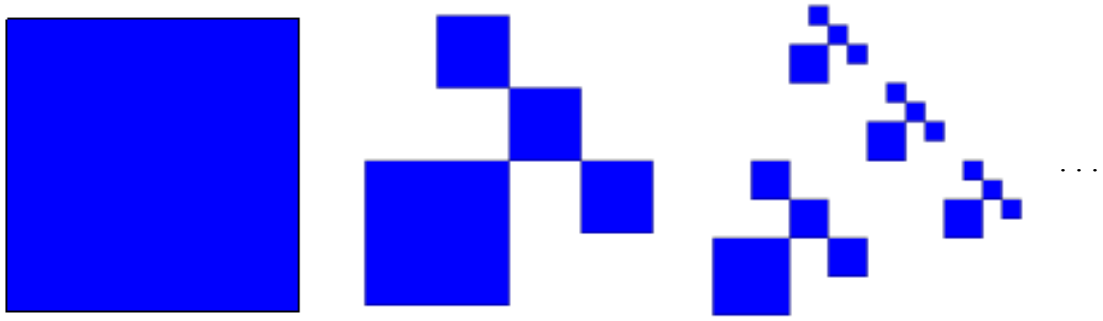


4.3.35 This sequence diverges since $\lim_{n\to\infty}\left(\frac{-3}{2}\right)^n \nexists$

Part II

Question 1 (8 points)

Consider the following sequence $\langle G_n \rangle$ of geometric figures, where G_0 is the unit square.



G_0 G_1 G_2 . . .

a) Find the dimension of the limiting figure (you can use Maple). Is it a fractal?

$$\left(\tfrac{1}{2}\right)^D + 3\left(\tfrac{1}{4}\right)^D = 1$$

$$D \approx 1.2$$

Since the figure is a self-similar figure with non-integer dimension, it is a fractal.

b) Give the recursive definition for the sequence $\langle G_n \rangle$.

$$f_1(x,y) = D_{\frac{1}{2}}(x,y) = \left(\tfrac{1}{2}x, \tfrac{1}{2}y\right)$$

$$f_2(x,y) = T_{\left(0,\frac{3}{4}\right)} \circ D_{\frac{1}{4}}(x,y) = \left(\tfrac{1}{4}x + \tfrac{1}{4}, \tfrac{1}{4}y + \tfrac{3}{4}\right)$$

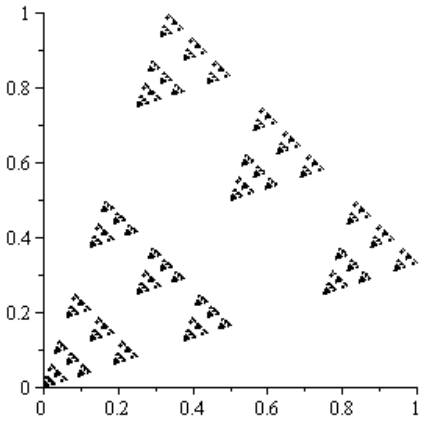
$$f_3(x,y) = T_{\left(\frac{1}{2},\frac{1}{2}\right)} \circ D_{\frac{1}{4}}(x,y) = \left(\tfrac{1}{4}x + \tfrac{1}{2}, \tfrac{1}{4}y + \tfrac{1}{2}\right)$$

$$f_4(x,y) = T_{\left(\frac{3}{4},0\right)} \circ D_{\frac{1}{4}}(x,y) = \left(\tfrac{1}{4}x + \tfrac{3}{4}, \tfrac{1}{4}y + \tfrac{1}{4}\right)$$

Thus $G_{n+1} = f_1(G_n) \cup f_2(G_n) \cup f_3(G_n) \cup f_4(G_n)$

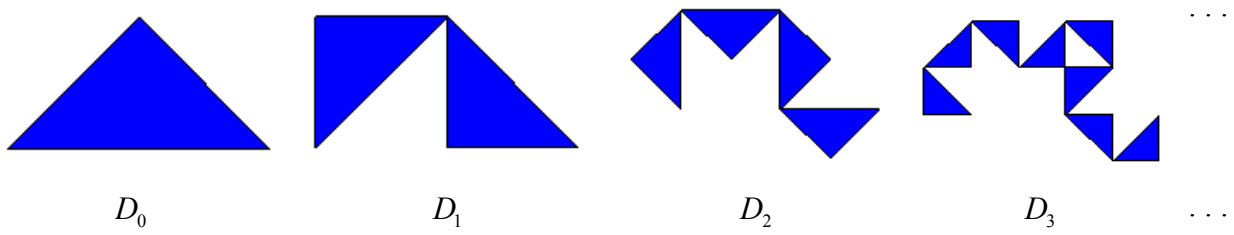
with $G_0 = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$, the filled unit square.

c) Use Maple to draw an approximation of G, where $\langle G_n \rangle \to G$.



Question 2 (6 points)

Consider the following sequence $\langle D_n \rangle$ of geometric figures, where D_0 is an isosceles right triangle, with base of length 1. The limiting figure, $\langle D_n \rangle \to D$ is called the *dragon fractal*.



D_0 D_1 D_2 D_3 . . .

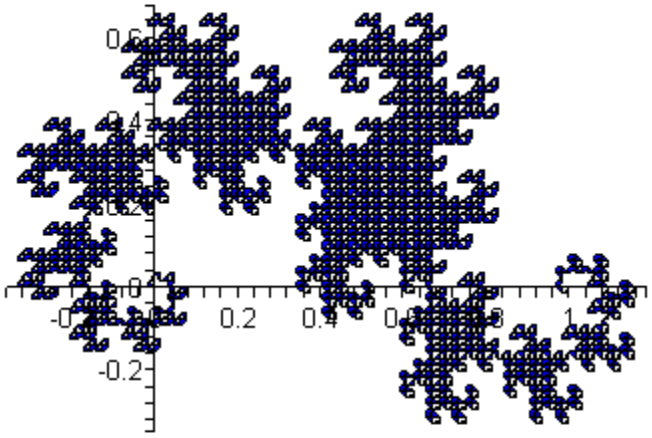
a) Give the recursive definition for the sequence $\langle D_n \rangle$.

$$f_1(x,y) = \left(R_{45^\circ} \circ D_{\frac{1}{\sqrt{2}}}\right)(x,y)$$
$$= \left(\tfrac{1}{\sqrt{2}}\cos 45^\circ x - \tfrac{1}{\sqrt{2}}\sin 45^\circ y, \tfrac{1}{\sqrt{2}}\sin 45^\circ x + \tfrac{1}{\sqrt{2}}\cos 45^\circ\right)$$
$$= \left(\tfrac{1}{2}x - \tfrac{1}{2}y, \tfrac{1}{2}x + \tfrac{1}{2}y\right)$$

$$f_2(x,y) = \left(T_{(1,0)} \circ R_{135^\circ} \circ D_{\frac{1}{\sqrt{2}}}\right)(x,y)$$
$$= \left(\tfrac{1}{\sqrt{2}}\cos 135^\circ x - \tfrac{1}{\sqrt{2}}\sin 135^\circ y + 1, \tfrac{1}{\sqrt{2}}\sin 135^\circ x + \tfrac{1}{\sqrt{2}}\cos 135^\circ\right)$$
$$= \left(-\tfrac{1}{2}x - \tfrac{1}{2}y + 1, \tfrac{1}{2}x - \tfrac{1}{2}y\right)$$

Thus $D_{n+1} = f_1(D_n) \cup f_2(D_n)$ with D_0 is the filled triangle ABC with vertices $A = (0,0)$, $B = (1,0)$ and $C = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$.

b) Use Maple to draw an approximation of D.



with 11 iterations