MATHEMATICS 201-BNJ-05
Topics in Mathematics
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Winter 2009

Assignment #1
SOLUTIONS

This assignment is due ***Monday February 2, 2009*** at the beginning of the class.

Part I

Do the following questions from the book.

1.2.50 page 11 (3 points)

$h_0 = 6$ $\quad h_{n+1} = \frac{2}{3} h_n$ $\quad$ geometric sequence with ratio $r = \frac{2}{3}$

$\langle h_n \rangle = \left\langle 6\left(\frac{2}{3}\right)^n \right\rangle_{n=0}^{\infty}$ $\quad h_{20} = 6\left(\frac{2}{3}\right)^{20} \simeq 0.0018$

Thus the ball will bounce 18 mm high after hitting the floor for the 20th time.

1.4.12 page 24 along with question 17 (for number 12) (9 points)

Fixed points: $B = \frac{1}{4} B\left(6 + B - B^2\right)$

$$B^3 - B^2 - 2B = 0$$
$$B\left(B^2 - B - 2\right) = 0$$
$$B\left(B-2\right)\left(B+1\right) = 0$$
$$B = -1, 0, 2$$

Hence the fixed points are $B = -1, 0, 2$

$$f\left(x\right) = \frac{1}{4} x\left(6 + x - x^2\right)$$
$$= \frac{3}{2}x + \frac{1}{4}x^2 - \frac{1}{4}x^3$$

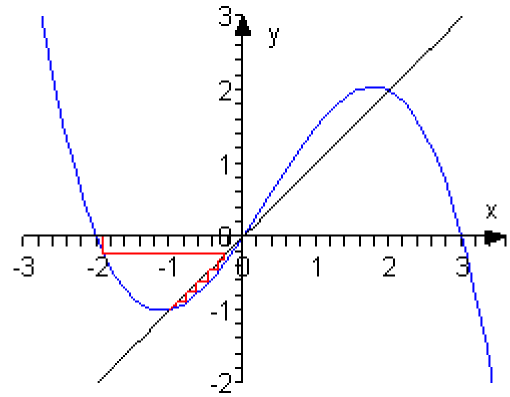
Intervals of $\nearrow / \searrow$: $f'\left(x\right) = \frac{3}{2} + \frac{1}{2}x - \frac{3}{4}x^2$

$$f'\left(x\right) = 0$$
$$x = \frac{1}{3} \pm \frac{1}{3}\sqrt{19}$$
$$\text{C.V.} \approx -1.12, 1.79$$

x	-∞	-1.12		1.79	∞
f'	-	0	+	0	-
f	↘	-1.02 min	↗	2.05 max	↘

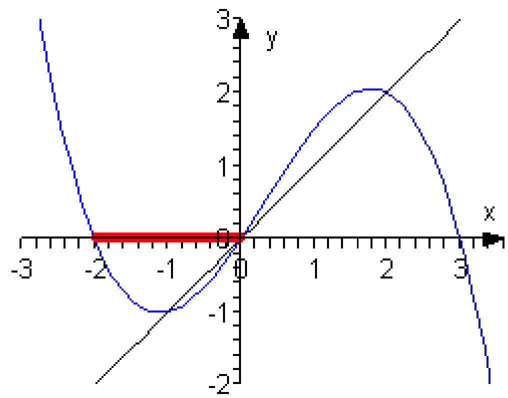
For $b_0 = -1.9$

$\langle b_n \rangle \to -1$



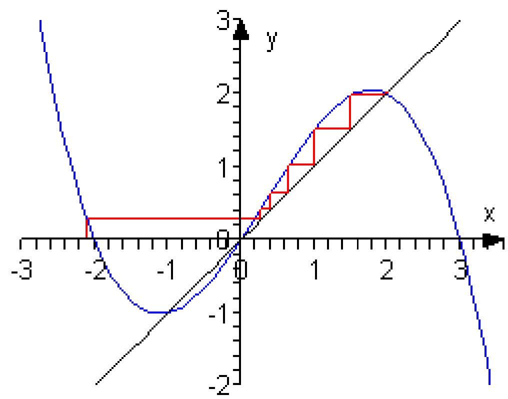
For $b_0 = -2$

$\langle b_n \rangle \to 0$



For $b_0 = -2.1$

$\langle b_n \rangle \to 2$



Since f' is continuous at -1 and $\left|f'(-1)\right| = \left|\frac{1}{4}\right| < 1$ then -1 is an attracting fixed point.

If $b_0 \in (-2,0)$, then $\langle b_n \rangle \to -1$ (see the above cobweb for $b_0 = -1.9$)

Since f' is continuous at 0 and $\left|f'(0)\right| = \left|\frac{3}{2}\right| > 1$ then 0 is a repelling fixed point

Since f' is continuous at 2 and $\left|f'(2)\right| = \left|\frac{-1}{2}\right| < 1$ then 2 is an attracting fixed point

If $b_0 \in (0,3)$, then $\langle b_n \rangle \to 2$ (see the above cobweb for $b_0 = -2.1$, after one iteration.)

1.5.4 page 29 (6 points)

a) $a_0 = 0$, $a_{n+1} = \frac{15}{40} a_n + 150$

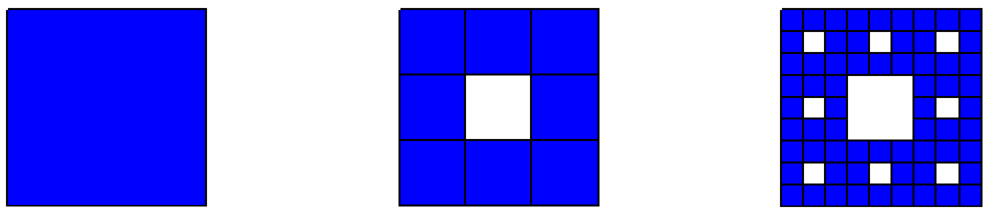
b) Fixed point $A = \frac{15}{40} A + 150$

$A = 240$

This is an affine system with $|r| = \left|\frac{3}{8}\right| < 1$, then $\langle a_n \rangle$ converges to its fixed point, hence $\langle a_n \rangle \to 240$, so in the long run, there will be 240 ml of additive

c) If $a_0 = 50$, then this does not effect convergence since $|r| = \left|\frac{3}{8}\right| < 1$, so in the long run, there will be 240 ml of additive

1.7.6 page 43 (2 points)



C_0 C_1 C_2

We have $N = 3$ and $s = \frac{1}{3}$, so the dimension is $D = \frac{\ln N}{\ln \frac{1}{s}} = \frac{\ln 8}{\ln 3} \simeq 1.89$.

Since the figure is self-similar with a non-integer dimension, then it is a fractal

Part II

Question 1 (13 points)

Consider the following number $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}$

a) Express this as a dynamical system $\langle a_n \rangle$.

$$a_{n+1} = \sqrt{2+a_n}$$

b) Find the fixed point(s) and determine whether it is attracting, repelling, or semi-repelling.

$$A = \sqrt{2+A} \qquad f(x) = \sqrt{2+x}$$

$$A^2 = 2 + A \qquad f'(x) = \frac{1}{2\sqrt{2+x}}$$

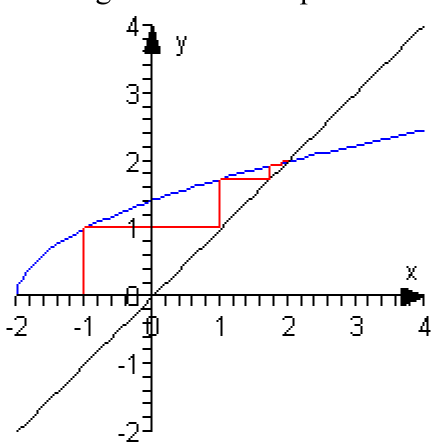
$$A^2 - A - 2 = 0$$

$$(A-2)(A+1) = 0$$

$$X = \cancel{-1}, 2$$

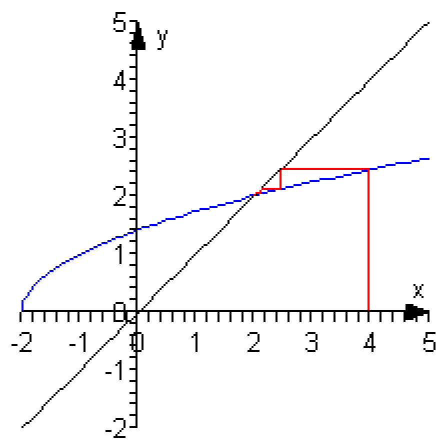
Since f' is continuous at 2 and $\left|f'(2)\right| = \frac{1}{4} < 1$, then 2 is an attracting fixed point.

c) Use a cobweb diagram to show that the dynamical system found in (a) has a solution that converges to the fixed point for almost any initial value.



$a_0 = -1$

$\langle a_n \rangle \to 2$



$a_0 = 4$

$\langle a_n \rangle \to 2$

Hence if $a_0 \geq -2$, then $\langle a_n \rangle \to 2$

d) Write a procedure that will give the n^{th} term of the dynamical system $\langle a_n \rangle$ (in decimal form!) if $a_1 = 1$.

e) Write a procedure that will give will list the first n terms of the dynamical system $\langle a_n \rangle$ (in decimal form!) if $a_1 = 1$.

Question 2 (9 points)
This problem is to be done entirely with Maple.
Consider the dynamical system $b_{n+1} = 2|b_n - 1|$.

a) Show that if $b_0 > 2$ or $b_0 < 0$ then the system diverges. (Show with cobweb diagrams made with Maple.)
b) Find a 3-cycle and illustrate with a cobweb diagram.
c) How many 3-cycles does the dynamical system have?
d) Find a 6-cycle and illustrate with a cobweb diagram.
e) Find a b_0 such that the sequence is chaotic. (Show with a cobweb diagram made with Maple).

Question 3 (4 points)
This problem is to be done entirely with Maple.
Consider the dynamical system given by $a_{n+1} = 3.2a_n - 2.2a_n^2$.

a) Find a 2-cycle and illustrate with a cobweb diagram.
b) Is the 2-cycle found in (a) is attracting or repulsing?

Question 4 (4 points)
Write a procedure in Maple that adds all the prime numbers between 1 and *n*.