MATHEMATICS 201-203-RE
Integral Calculus
Martin Huard
Winter 2009

XXIII – Taylor and Maclaurin Series

1. Find the Maclaurin Series for $f(x)$ using the definition. Give the radius of convergence.
   a) $f(x)=\cos x$
   b) $f(x)=\sin 2x$
   c) $f(x)=\ln(1+x)$
   d) $f(x)=(1+x)^{-4}$
   e) $f(x)=\sqrt[4]{1+x}$
   f) $f(x)=\dfrac{e^x+e^{-x}}{2}$

2. Find the Taylor series for $f(x)$ at $x=a$.
   a) $f(x)=x^3-3x+2 \quad a=5$
   b) $f(x)=\sqrt{x} \quad a=9$
   c) $f(x)=\sin x \quad a=\frac{\pi}{2}$
   d) $f(x)=\cos x \quad a=-\frac{\pi}{4}$
   e) $f(x)=\ln x \quad a=3$
   f) $f(x)=\dfrac{1}{\sqrt[3]{x}} \quad a=1$

3. Approximate f by a Taylor polynomial with degree n at the number a. Use this approximation to estimate the given number, and estimate the accuracy of that approximation.
   a) $f(x)=\sqrt{1+x} \quad n=1 \quad a=0 \quad \sqrt{1.1}$
   b) $f(x)=\dfrac{1}{\sqrt[3]{x+2}} \quad n=3 \quad a=6 \quad \frac{1}{\sqrt[3]{7.9}}$
   c) $f(x)=\cos^3 x \quad n=2 \quad a=\frac{\pi}{6} \quad \cos^3 31°$
   d) $f(x)=\ln x \quad n=5 \quad a=1 \quad \ln 1.5$
   e) $f(x)=e^{x^2} \quad n=3 \quad a=0 \quad e^{0.01}$

4. Using the Maclaurin series for e^x, find the Maclaurin for the given functions.
   a) $f(x)=e^{4x}$
   b) $f(x)=x^3e^{-x}$
   c) $f(x)=e^{x^2}$
   d) $f(x)=\dfrac{e^x+e^{-x}}{2}$

5. Using the Maclaurin series for $\sin x$ or $\cos x$, find the Maclaurin for the given functions.
   a) $f(x)=x\sin 3x$
   b) $f(x)=\cos\left(\frac{x}{2}\right)$

6. Use multiplication or division of known power series to find the first three nonzero terms in the Maclaurin series for each function.

   a) $f(x) = e^{-x^2}\cos x$ b) $f(x) = \sec x$

7. Using the Maclaurin series for $\sin x$, $\cos x$ or e^x, evaluate the indefinite integral as an infinite series.

   a) $\int \frac{\sin 2x}{x}\,dx$ b) $\int x^2 e^{-x^2}\,dx$ c) $\int \sin\left(x^4\right)dx$

8. Using the Maclaurin series for $\sin x$, $\cos x$ or e^x, approximate the definite integral to within five decimal places.

   a) $\int_0^{\frac{1}{2}} \cos\left(x^2\right)dx$ b) $\int_0^1 \frac{\sin x}{x}\,dx$ c) $\int_0^1 \sin\left(x^3\right)dx$

9. Consider the function $f(x) = \sqrt[3]{3x+1}$.

   a) Find the Maclaurin series for $f(x)$.

   b) Use the answer from (a) to approximate the definite integral $\int_0^{\frac{1}{3}} \sqrt[3]{3x+1}$ to within five decimal places.

10. Find the sum of the series.

   a) $\sum_{n=0}^{\infty} \frac{2^n}{7^n n!}$ b) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$ c) $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$

Answers

1. a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $R = \infty$

   b) $\sin 2x = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$ $R = \infty$

   c) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$ $R = 1$

d) $\left(1+x\right)^{-4}=1-4x+\frac{4\cdot5x^2}{2!}-\frac{4\cdot5\cdot6x^3}{3!}+\frac{4\cdot5\cdot6\cdot7x^4}{4!}-\frac{4\cdot5\cdot6\cdot7\cdot8x^4}{5!}+\ldots=\sum_{n=3}^{\infty}\frac{n(n-1)(n-2)(-1)^{n-1}}{6}x^{n-3}$ $\quad R=1$

e) $\sqrt[4]{1+x}=1+\frac{1}{4}x-\frac{3}{4^2 2!}x^2+\frac{3\cdot7}{4^3 3!}x^3-\frac{3\cdot7\cdot11}{4^4 4!}x^4+\cdots=1+\frac{1}{4}x+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}3\cdot7\cdot11\cdot\cdots\cdot(4n-5)x^n}{4^n n!}$ $\quad R=1$

f) $\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ $\quad R=\infty$

2. a) $x^3-3x+2=112+72\left(x-5\right)+15\left(x-5\right)^2+\left(x-5\right)^3$

b) $\sqrt{x}=3+\frac{1}{6}\left(x-9\right)-\frac{1}{216}\left(x-9\right)^2+\frac{1}{3888}\left(x-9\right)^3-\ldots=3+\frac{1}{6}\left(x-9\right)+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot3\cdot5\cdots(2n-3)}{3^{2n-1}2^{n+2}n!}\left(x-9\right)^n$

c) $\sin x=1-\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$

d) $\cos x=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right)-\frac{\sqrt{2}}{2\cdot2!}\left(x+\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{2\cdot3!}\left(x+\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{2\cdot4!}\left(x+\frac{\pi}{4}\right)^4+\ldots$

e) $\ln x=\ln3+\frac{1}{3}\left(x-3\right)-\frac{1}{9\cdot2}\left(x-3\right)^2+\frac{1}{27\cdot3}\left(x-3\right)^3-\ldots=\ln3+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-3)^n}{n3^n}$

f) $\frac{1}{\sqrt[3]{x}}=1-\frac{1}{3}\left(x-1\right)+\frac{4}{3^2\cdot2!}\left(x-1\right)^2-\frac{4\cdot7}{3^3 3!}\left(x-1\right)^3+\frac{4\cdot7\cdot10}{3^4 4!}\left(x-1\right)^4-\cdots=1+\sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot4\cdot7\cdot10\cdot\cdots\cdot(3n-2)(x-1)^n}{3^n n!}$

3. a) $\sqrt{1+x}\approx1+\frac{1}{2}x$ $\qquad \sqrt{1.1}\approx\frac{21}{20}=1.05$ $\qquad \left|R_1\left(x\right)\right|<0.0011$

b) $\frac{1}{\sqrt[3]{x+2}}=\frac{1}{2}-\frac{1}{48}\left(x-6\right)+\frac{1}{576}\left(x-6\right)^2-\frac{7}{41472}\left(x-6\right)^3$

$\frac{1}{\sqrt[3]{7.9}}\approx\frac{20823127}{4147200}\simeq0.5021009$ $\qquad \left|R_3\left(x\right)\right|<0.000\,000\,002$

c) $\cos^3 x\approx\frac{3\sqrt{3}}{8}-\frac{9}{8}\left(x-\frac{\pi}{6}\right)-\frac{3\sqrt{3}}{16}\left(x-\frac{\pi}{6}\right)^2$ $\qquad \cos^3 31^\circ\approx0.629785$ $\qquad \left|R_2\left(x\right)\right|<0.000024$

d) $\ln\left(x\right)\approx\left(x-1\right)-\frac{1}{2}\left(x-1\right)^2+\frac{1}{3}\left(x-1\right)^3-\frac{1}{4}\left(x-1\right)^4+\frac{1}{5}\left(x-1\right)^5$

$\ln\left(2\right)\approx\frac{391}{960}\simeq0.407292$ $\qquad \left|R_5\left(x\right)\right|\le\frac{1}{6}\simeq0.167$

e) $e^{x^2}\approx1+x^2$ $\qquad e^{0.01}\approx\frac{101}{100}=1.01$ $\qquad \left|R_3\left(x\right)\right|\le\frac{1}{20000}=0.00005$

4. a) $e^{4x}=\sum_{n=0}^{\infty}\frac{4^n x^n}{n!}$ $\qquad$ b) $x^3e^{-x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{n!}$ $\qquad$ c) $e^{x^2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$ $\qquad$ d) $\frac{e^x+e^{-x}}{2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

5. a) $x\sin3x=\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n+1}x^{2n+2}}{(2n+1)!}$ $\qquad$ b) $\cos\left(\frac{x}{2}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n(2n)!}$

6. a) $e^{-x^2}\cos x=1-\frac{3}{2}x^2+\frac{25}{24}x^4-\cdots$ $\qquad$ b) $\sec x=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots$

7. a) $\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}x^{2n+1}}{(2n+1)(2n+1)!}+C$ $\qquad$ b) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+3)n!}+C$ $\qquad$ c) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+5}}{(8n+5)(2n+1)!}+C$

8. a) 0.49688 $\qquad$ b) 0. 94608 $\qquad$ c) 0.23385

9. a) $\sqrt[3]{3x+1}=1+x-\frac{2}{2!}x^2+\frac{2\cdot5}{3!}x^3-\frac{2\cdot5\cdot8}{4!}x^4+\ldots=1+x+\sum_{n=1}^{\infty}\frac{(-1)^n 2\cdot5\cdot8\cdots(3n-1)}{(n+1)!}x^{n+1}$ $\qquad$ b) 0.37996

10. a) $e^{\frac{2}{7}}$ $\qquad$ b) $\frac{\sqrt{3}}{2}$ $\qquad$ c) $\frac{1}{2}$