MATHEMATICS 201-203-RE
Integral Calculus
Martin Huard
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XXII – Power Series

1. Find the radius and interval of convergence.

a) $\sum_{k=1}^{\infty} 3^k x^k$ b) $\sum_{n=1}^{\infty} \frac{x^n}{2^n\sqrt{n}}$ c) $\sum_{k=1}^{\infty} \frac{3^k x^{2k}}{k+2}$

d) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ e) $\sum_{n=1}^{\infty} \frac{nx^n}{5^n}$ f) $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{\ln n}$

g) $\sum_{k=1}^{\infty} \frac{k!(x-3)^k}{2^k}$ h) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k(x+5)^{k-1}}{9^{k-1}}$ i) $\sum_{k=1}^{\infty} \frac{2^k (x+1)^k}{\ln(k+1)}$

j) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{2n(2n-1)}$ k) $\sum_{k=1}^{\infty} \frac{(-1)^k 2^k (x+2)^k}{(k+1)^3}$ l) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)!}$

m) $\sum_{k=1}^{\infty} \frac{1}{k}(3x-2)^k$ n) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{(n+1)!}(x+3)^n$ o) $\sum_{k=1}^{\infty} \left(\frac{k}{k-1}\right)\frac{(x+2)^k}{2^k}$

p) $1-(x+3)+(x+3)^2-(x+3)^3+\ldots$ q) $1-\frac{2x}{2}+\frac{3x^2}{4}-\frac{4x^3}{8}+\frac{5x^4}{16}-\ldots$

r) $\frac{3(x+1)^2}{4}+\frac{9(x+1)^4}{9}+\frac{27(x+1)^6}{16}+\frac{81(x+1)^8}{25}+\ldots$ s) $\sum_{k=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2k)}{1\cdot 3\cdot 5\cdots(2k-1)} x^k$

2. Use the power series for $g(x)=\frac{1}{1-x}$ to find a power series representation for the given function. Determine the radius of convergence.

a) $f(x)=\frac{1}{1+x}$ b) $f(x)=\frac{1}{1-x^3}$ c) $f(x)=\frac{1}{1+2x}$

d) $f(x)=\frac{1}{2-x}$ e) $f(x)=\frac{x^3}{1+x^2}$

3. Using the power series for $f(x)=\frac{1}{1+x}$ found in question 2,

a) use differentiation to find a power series for $\frac{1}{(1+x)^2}$

b) use differentiation to find a power series for $\frac{1}{(1+x)^3}$

c) find a power series for $\frac{1}{1+x^2}$

d) use differentiation along with the answer in (c) to find the power series for $\frac{x}{(1+x^2)^2}$.

4. Consider the power series for $f(x)=\frac{1}{1+x}$ found in question 2.
   a) Use integration to find a power series for $\ln(1+x)$.
   b) Use the power series found in (a) to find the power series for $\ln(1+x^4)$.
   c) Use the power series found in (b) to estimate the value of $\int_0^{\frac{1}{2}} \ln(1+x^4)\,dx$ to 5 decimal places.
   d) Find the power series for $\frac{1}{9+x^2}$
   e) Use integration along with the answer in (d) to find the power series for $\arctan\frac{x}{3}$.
   f) Use the power series found in (e) to estimate the value of $\int_0^{\frac{1}{3}} \arctan\frac{x}{3}\,dx$ to 5 decimal places.

5. Consider the function $f(x)=\frac{x}{x^2-x-6}$.
   a) Find the power series for *f* using the sum of two power series. Also, give the first five nonzero terms.
   $$\frac{x}{x^2-x-6}=\frac{\frac{3}{5}}{x-3}+\frac{\frac{2}{5}}{x+2}$$
   b) Find the first five nonzero terms for the power series of *f* using the product of two power series.
   $$\frac{x}{x^2-x-6}=\left(\frac{x}{x-3}\right)\left(\frac{1}{x+2}\right)$$
   c) Find the first five nonzero terms for the power series of *f* using long division.

Answers

1. a) $R=\frac{1}{3},\ I=\left(-\frac{1}{3},\frac{1}{3}\right)$ b) $R=2,\ I=[-2,2)$ c) $R=\frac{\sqrt{3}}{3},\ I=\left[-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)$
   d) $R=\infty,\ I=(-\infty,\infty)$ e) $R=5,\ I=(-5,5)$ f) $R=1,\ I=(-1,1]$
   g) $R=0,\ I=\{3\}$ h) $R=9,\ I=(-14,4)$ i) $R=\frac{1}{2},\ I=\left[-\frac{3}{2},-\frac{1}{2}\right)$
   j) $R=1,\ I=[-2,0]$ k) $R=\frac{1}{2},\ I=\left[-\frac{5}{2},-\frac{3}{2}\right]$ l) $R=\infty,\ I=(-\infty,\infty)$
   m) $R=\frac{1}{3},\ I=\left[\frac{1}{3},1\right)$ n) $R=\infty,\ I=(-\infty,\infty)$ o) $R=2,\ I=(-4,0)$
   p) $R=1,\ I=(-4,-2)$ q) $R=2,\ I=(-2,2)$ r) $R=\frac{1}{3},\ I=\left[-\frac{4}{3},-\frac{2}{3}\right]$
   s) $R=1,\ I=(-1,1)$

2. a) $\sum_{n=0}^{\infty}(-1)^n x^n \quad R=1$ b) $\sum_{n=0}^{\infty}x^{3n} \quad R=1$ c) $\sum_{n=0}^{\infty}(-1)^n 2^n x^n \quad R=\frac{1}{2}$
   d) $\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}} \quad R=2$ e) $\sum_{n=0}^{\infty}(-1)^n x^{2n+3} \quad R=1$

3. a) $\sum_{n=1}^{\infty}(-1)^{n+1} nx^{n-1}$ b) $\sum_{n=2}^{\infty}\frac{1}{2}n(n-1)(-1)^n x^{n-2}$ c) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$
   d) $\sum_{n=1}^{\infty}(-1)^{n+1} nx^{2n-1}$

4. a) $\sum_{n=0}^{\infty}\frac{1}{n}(-1)^{n+1} x^n$ b) $\sum_{n=0}^{\infty}\frac{1}{n}(-1)^{n+1} x^{4n}$
   c) $\int_0^{\frac{1}{2}}\ln\left(1+x^4\right)dx=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(4n+1)2^{4n+1}}\simeq\frac{1}{160}-\frac{1}{9216}+\frac{1}{319488}\simeq 0.00614$
   d) $\sum_{n=0}^{\infty}\frac{1}{9^{n+1}}(-1)^n x^{2n}$ e) $\sum_{n=0}^{\infty}\frac{1}{(2n+1)3^{2n+1}}(-1)^n x^{2n+1}$
   f) $\int_0^{\frac{1}{3}}\arctan\frac{x}{3}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+2)3^{4n+3}}\simeq\frac{1}{54}-\frac{1}{26244}+\frac{1}{5314410}\simeq 0.01848$

5. $\sum_{n=0}^{\infty}\frac{\left((-3)^n-2^n\right)x^n}{5\cdot 6^n}=-\frac{1}{6}x+\frac{1}{36}x^2-\frac{7}{216}x^3+\frac{13}{1296}x^4-\frac{55}{7776}x^5+\cdots$