**MATHEMATICS 201-203-RE**
Integral Calculus
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Winter 2009

# XVII – Differential Equations

1. Solve the following differential equation.
   a) $\frac{dy}{dx}=\left(1+y^2\right)x^2$
   b) $\frac{dy}{dx}=y\sin\left(2x+3\right)$
   c) $\frac{dy}{dx}=\frac{xy}{2\ln y}$
   d) $\left(1+x^4\right)\frac{dy}{dx}=\frac{x^3}{y}$
   e) $\frac{dy}{dx}=xy^2-x-y^2+1$
   f) $e^{-y}\sin x-y'\cos^2 x=0$
   g) $\frac{dy}{dt}=\frac{te^t}{y\sqrt{1+y^2}}$
   h) $\frac{dz}{dt}+e^{t+z}=0$
   i) $\frac{dy}{dx}-\frac{y^2-y}{\cos x}=0$
   j) $e^y\sin x\,dx+\cos x\left(e^{2y}-y\right)dy=0$

2. Find the solution of the differential equation that satisfies the given initial condition.
   a) $\frac{dy}{dx}=\frac{4x^2}{y+\cos y}$, $\quad y\left(1\right)=\pi$
   b) $\frac{dy}{dx}=\frac{2x+\sec^2 x}{2y}$, $\quad y\left(0\right)=-5$
   c) $\frac{dy}{dx}=\frac{x^2y-y}{y+1}$, $\quad y\left(3\right)=1$
   d) $xe^{-t}\frac{dx}{dt}=t$, $\quad x\left(0\right)=1$
   e) $\frac{dy}{dx}=y^2-3y-10$, $\quad y\left(0\right)=6$
   f) $x+2y\sqrt{x^2+1}\frac{dy}{dx}=0$, $\quad y\left(0\right)=1$
   g) $\frac{dx}{dt}=\frac{e^{t-x}}{1+e^t}$, $\quad x\left(1\right)=0$
   h) $\frac{dy}{dx}=x\sqrt{\frac{1-y}{1-x^2}}$, $\quad y\left(0\right)=0$

3. A certain small country has \$10 billion in paper currency in circulation, and each day \$50 million comes into the country's banks. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks. This is described by the differential equation
$$\frac{dx}{dt}=\frac{10-x}{10}\cdot 0.005$$
where $x$ denotes the amount of new currency (in billions of \$) in circulation at time $t$ (in days).
   a) Solve this differential equation, assuming that at $t=0$, $x=0$.
   b) How long will it take for the new bills to account for 90% of the currency in circulation?

4. The management of a factory has found that a worker can produce at most 40 units per day. The number of units $N$ per day produced by a new employee will increase at a rate proportional to the difference between 40 and $N$. This is described by the differential equation $\frac{dN}{dt}=k\left(40-N\right)$, where $t$ is the time in days. Solve this differential equation.

5. The rate of increase in sales $S$ (in thousands of units) of a product is proportional to the current level of sales and inversely proportional to the square of the time $t$. This is described by the differential equation
$$\frac{dS}{dt}=k\frac{S}{t^2}$$
where $t$ is the time in years.
   a) Solve this differential equation.
   b) The saturation point for the market is 50 000 units. That is $\lim_{t\to\infty} S=50$. Also, after 1 year, 10 000 units have been sold. Find $S$ as a function of the time $t$.

6. According to the economist Vilfredo Pareto (1848-1923), the rate of decrease of the number of people $y$ in a stable economy having an income of at least $x$ dollars is directly proportional to the number of such people and inversely proportional to their income $x$. This is modeled by the differential equation
$$\frac{dy}{dx}=-k\frac{y}{x}$$
Solve this differential equation.

7. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction $y$ of the population who has heard the rumor and the fraction who have not heard the rumor. This gives rise to the differential equation
$$\frac{dy}{dt}=ky\left(1-y\right)$$
   a) Solve the differential equation.
   b) A small town has 1000 inhabitants. At 8 A.M., 80 people have heard the rumor and by noon half the town has heard it. At what time will 90% of the population have heard the rumor?

8. Suppose that in a certain company the relationship between the price per unit $p$ of their product and the weekly sales volume $y$ in thousands of dollars is given by
$$\frac{dy}{dp}=-\frac{2}{5}\left(\frac{y}{p+8}\right)$$
Solve this differential equation if $y=8$ when $p$ = \$24.

9. The growth of a population can sometimes be modeled with the Gompertz differential equation, given by

$$\frac{dP}{dt} = kP\ln\tfrac{M}{P} \qquad \text{or} \qquad P = Me^{Ae^{-kt}}\ \frac{dP}{dt} = kP\left(\ln M - \ln P\right)$$

where $M$ is the maximal carrying capacity.
   a) Solve this differential equation.
   b) A biologist stocked a lake with 400 fish, and estimates that the maximum population the lake can sustain is 10000. If the number of fish tripled in the first year, what will be the population after $t$ years?

10. Sarah estimates that the elasticity of demand for her maple croissants is $E = 0.05p - 1.5$ where $p$ is the price per croissant. She sells 35 maple croissants per day when the price is \$2.00 per croissant. Find a formula expressing the demand $x$ as a function of $p$. Recall that the elasticity of demand is given by

$$E = -\frac{p}{x}\frac{dx}{dp}$$

11. A new model of DVD player has just been introduced in the market. A market saturation of 3 000 000 DVD players is predicted, and the total sales will be governed by the equation

$$\frac{dS}{dt} = \tfrac{1}{6}S\left(3 - S\right)$$

where $S$ is the total sales in millions of DVD players and $t$ is measured in months. If 1000 DVD players are given away when the new model is introduced, what will the sales be $t$ months from now? How long will it take for the sales to reach 2 000 000 DVD players?

12. Payments are made continuously on a mortgage of \$200 000 at the constant rate of \$1500 per month. Let $P$ denote the balance (amount still owned) after $t$ months, and suppose the annual interest rate is 6% compounded month. This gives the following differential equation:

$$\frac{dP}{dt} = \tfrac{0.06}{12}P - 1500$$

   a) Solve this differential equation.
   b) How long will it take to pay off the mortgage?

13. Question 12 can be generalized to a mortgage of $P_0$ dollars with payments at the constant rate of $R$ dollars per month, with a monthly interest rate $r$%. This gives the following differential equation:

$$\frac{dP}{dt} = rP - R$$

   a) Solve this differential equation.
   b) If a car loan of \$4800 must be paid off continuously in 3 years at an annual interest rate of 9%, what will be the monthly payments?

# Answers

1. a) $y=\tan\left(\frac{x^3}{3}+C\right)$ b) $y=Ce^{-\frac{1}{2}\cos(2x+3)}$

   c) $(\ln y)^2=\frac{1}{2}x^2+C$ d) $y^2=\frac{1}{2}\ln\left(1+x^4\right)+C$

   e) $\ln\left|\frac{y-1}{y+1}\right|=x^2-2x+C$ or $y=\frac{1+e^{x^2-2x+C}}{1-e^{x^2-2x+C}}$

   f) $y=\ln(\sec x+C)$ g) $\left(1+y^2\right)^{\frac{3}{2}}=3te^t-3e^t+C$

   h) $z=-\ln\left(e^t+C\right)$ i) $\frac{y-1}{y}=C\sec x+C\tan x$ or $y=\frac{\cos x}{\cos x-C(1+\sin x)}$

   j) $e^y+ye^{-y}+e^{-y}=\ln|\sec x|+C$

2. a) $y^2+2\sin y=\frac{8}{3}x^3+\pi^2-\frac{8}{3}$ b) $y^2=x^2+\tan x+25$

   c) $3y+3\ln|y|=x^3-3x-15$ d) $x^2=2te^t-2e^t+3$

   e) $\frac{y-5}{y+2}=\frac{1}{8}e^{7x}$ or $y=\frac{40+2e^{7x}}{8-e^{7x}}$ f) $y^2=-\sqrt{x^2+1}+2$

   g) $e^x=\ln\left(1+e^t\right)+1-\ln(1+e)$ h) $y=1-\frac{1}{4}\left(1+\sqrt{1-x^2}\right)^2$

3. a) $x=10\left(1-e^{-0.005t}\right)$ b) 12.6 years

4. $N=40-Ce^{-kt}$

5. a) $S=Ce^{-\frac{k}{t}}$ b) $S=50\left(\frac{1}{5}\right)^{\frac{1}{t}}$

6. $y=Cx^{-k}$

7. a) $\frac{y}{1-y}=Ce^{kt}$, or $y=\frac{Ce^{kt}}{1+Ce^{kt}}=\frac{1}{1+Ae^{-kt}}$ b) $\frac{y}{1-y}=\frac{2}{23}\left(\frac{23}{2}\right)^{\frac{t}{4}}$ or $y=\frac{2}{2+23\left(\frac{2}{23}\right)^{\frac{t}{4}}}$, 3:36 P.M.

8. $y=\frac{32}{(p+8)^{\frac{2}{5}}}$

9. a) $P(t)=e^{M-Ae^{-kt}}$ b) $P(t)=10000\left(\frac{1}{25}\right)^{\left(\frac{\ln\frac{3}{25}}{\ln\frac{1}{25}}\right)^t}\simeq 10000(0.04)^{0.6587^t}$

10. $x=\frac{35\sqrt{2}}{4}p^{\frac{3}{2}}e^{\frac{-p}{20}+\frac{1}{10}}$

11. $S(t)=\frac{3}{1+299e^{\frac{-t}{2}}}$ It will take $t=2\ln 598\simeq 12.8$ months

12. a) $P(t)=300000-280000e^{\frac{t}{200}}$ b) $t=200\ln 3\simeq 220$ months or 18 years and 4 months

13. a) $P(t)=\frac{R}{r}+\left(P_0-\frac{R}{r}\right)e^{rt}$ b) \$152.14