MATHEMATICS 201-NYB-05
Integral Calculus
Martin Huard
Winter 2008

VI - Integration by Parts

1. Evaluate the integral.
   a) $\int xe^{-x}\,dx$
   b) $\int x^2e^x\,dx$
   c) $\int x\cos 3x\,dx$
   d) $\int x^2\sin x\,dx$
   e) $\int \sqrt{x}\ln x\,dx$
   f) $\int (\ln x)^2\,dx$
   g) $\int \frac{xe^x}{(1+x)^2}dx$
   h) $\int e^{2x}\cos 3x\,dx$
   i) $\int e^{ax}\sin bx\,dx$
   j) $\int \cos(\ln x)\,dx$
   k) $\int \frac{\ln x}{x^2}dx$
   l) $\int \sin 3\theta\cos 5\theta\,d\theta$
   m) $\int \arcsin 2x\,dx$
   n) $\int e^{\sqrt{t}}dt$
   o) $\int x\tan^2 x\,dx$

2. Evaluate the given integral.
   a) $\int_0^1 xe^{2x}\,dx$
   b) $\int_1^{e^2} x\ln\sqrt{x}\,dx$
   c) $\int_0^{\frac{\pi}{9}} 4x^2\sin 3x\,dx$
   d) $\int_0^2 \frac{x+2}{e^{3x}}dx$
   e) $\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx$
   f) $\int_{-1}^1 \arccos x\,dx$
   g) $\int_{11}^{18} r\sqrt{r-2}\,dr$
   h) $\int_0^1 x\arctan\left(x^2\right)dx$
   i) $\int_2^4 \operatorname{arcsec}\sqrt{x}\,dx$

3. Prove the following reduction formulas.
   a) $\int x^ne^xdx = x^ne^x - n\int x^{n-1}e^xdx$
   b) $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$
   c) $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$
   d) $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$

4. Use the reduction formulas proved in Question 3 to evaluate the following integrals.
   a) $\int x^6e^xdx$
   b) $\int (\ln x)^4\,dx$
   c) $\int \cos^5 x\,dx$
   d) $\int_0^{\frac{\pi}{4}} \tan^6 x\,dx$

5. Prove that for even powers of Cosine, $\int_0^{\frac{\pi}{2}} \cos^{2n}x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 5\cdot\cdots\cdot 2n}\frac{\pi}{2}$.

6. Prove that if $f(0)=g(0)=0$ and f'' and g'' are continuous, then
$$\int_0^t f(x)g''(x)dx = f(t)g'(t) - f'(t)g(t) + \int_0^t f''(x)g(x)dx$$

Answers

1. a) $-xe^{-x}-e^{-x}+C$ b) $x^2e^x-2xe^x+2e^x+C$
   c) $\frac{1}{3}x\sin(3x)+\frac{1}{9}\cos(3x)+C$ d) $-x^2\cos x+2x\sin x+2\cos x+C$
   e) $\frac{2}{3}x^{\frac{3}{2}}\ln x-\frac{4}{9}x^{\frac{3}{2}}+C$ f) $x(\ln x)^2-2x\ln x+2x+C$
   g) $\frac{e^x}{1+x}+C$ h) $\frac{2}{13}e^{2x}\cos 3x+\frac{3}{13}e^{2x}\sin 3x+C$
   i) $\frac{a}{a^2+b^2}e^{ax}\sin bx-\frac{b}{a^2+b^2}e^{ax}\cos bx+C$ j) $\frac{1}{2}x\cos(\ln x)+\frac{1}{2}x\sin(\ln x)+C$
   k) $-\frac{\ln x}{x}-\frac{1}{x}+C$ l) $\frac{5}{16}\sin 3\theta\sin 5\theta+\frac{3}{16}\cos 3\theta\cos 5\theta+C$
   m) $x\arcsin(2x)+\frac{1}{2}\sqrt{1-4x^2}+C$ n) $2\sqrt{t}e^{\sqrt{t}}-2e^{\sqrt{t}}+C$
   o) $x\tan x-\frac{1}{2}x^2+\ln|\cos x|+C$
2. a) $\frac{1}{4}e^2+\frac{1}{4}$ b) $\frac{3}{8}e^4+\frac{1}{8}$ c) $-\frac{2}{243}\pi^2-\frac{4}{27}+\frac{4}{81}\pi\sqrt{3}$
   d) $\frac{7}{9}-\frac{13}{9e^6}$ e) $\frac{1}{16}\pi^2+\frac{1}{4}$ f) π
   g) $\frac{5426}{15}$ h) $\frac{\pi}{8}-\frac{1}{4}\ln 2$ i) $\frac{5\pi}{6}-\sqrt{3}+1$
4. a) $\left(x^6-6x^5+30x^4-120x^3+360x^2-720x+720\right)e^x+C$
   b) $x(\ln x)^4-4x(\ln x)^3+12x(\ln x)^2-24x\ln x+24x+C$
   c) $\frac{1}{5}\cos^4 x\sin x+\frac{4}{15}\cos^2 x\sin x+\frac{8}{15}\sin x+C$
   d) $\frac{13}{15}-\frac{\pi}{4}$
5. $$\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx=\left[\frac{1}{2n}\cos^{2n-1}x\sin x\right]_0^{\frac{\pi}{2}}+\frac{2n-1}{2n}\int_0^{\frac{\pi}{2}}\cos^{2n-2}x\,dx$$ applying reduction formula
$$=(0-0)+\frac{2n-1}{2n}\left[\frac{1}{2n-2}\cos^{2n-3}x\sin x\right]_0^{\frac{\pi}{2}}+\frac{2n-3}{2n-2}\int_0^{\frac{\pi}{2}}\cos^{2n-4}x\,dx$$
$$=\frac{(2n-1)(2n-3)}{2n(2n-2)}\int_0^{\frac{\pi}{2}}\cos^{2n-4}x\,dx$$
$$\ldots$$
$$=\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 5\cdot\cdots\cdot 2n}\int_0^{\frac{\pi}{2}}dx$$
$$==\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 5\cdot\cdots\cdot 2n}\frac{\pi}{2}$$
6. $$\int_0^t f(x)g''(x)\,dx=\left[f(x)g'(x)\right]_0^t-\int_0^t f'(x)g'(x)\,dx$$
$$=f(t)g'(t)-\left[f'(x)g(x)\right]_0^t+\int_0^t f''(x)g(x)\,dx$$
$$=f(t)g'(t)-f'(t)g(t)+\int_0^t f''(x)g(x)\,dx$$

$u=f(x)$ $v=g'(x)$
$du=f'(x)dx$ $dv=g''(x)dx$

$u=f'(x)$ $v=g(x)$
$du=f''(x)dx$ $dv=g'(x)dx$