

MATHEMATICS 201-203-RE

Integral Calculus

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Test #4
SOLUTIONS

Question 1 (6 points)

Consider the sequence $\left\langle 3 - \frac{1}{n^2} \right\rangle_{n=1}^{\infty}$.

- a) Determine whether the sequence is increasing, decreasing or nonmonotonic.

$$f(x) = 3 - \frac{1}{x^2} \quad f'(x) = -(-2)(x^{-3}) = \frac{2}{x^3} > 0 \quad \text{if } x \geq 1$$

Thus the sequence $\left\langle 3 - \frac{1}{n^2} \right\rangle_{n=1}^{\infty}$ is increasing.

- b) Determine if the sequence is bounded. If so, find upper and lower bounds.

Since $\left\langle 3 - \frac{1}{n^2} \right\rangle$ is increasing, then the lower bound can be found at $n = 1$.

Since $\lim_{n \rightarrow \infty} \left(3 - \frac{1}{n^2} \right) = 3$, then 3 is an upper bound.

Hence, $\left\langle 3 - \frac{1}{n^2} \right\rangle_{n=1}^{\infty}$ is bounded below by 2 and bounded above by 3

$$2 \leq a_n \leq 3$$

- c) Determine if the sequence converges. If so, to what value?

$$\lim_{n \rightarrow \infty} \left(3 - \frac{1}{n^2} \right) = 3 \quad \text{thus } \left\langle 3 - \frac{1}{n^2} \right\rangle_{n=1}^{\infty} \text{ converges to } 3$$

Question 2 (10 points)

Find the sum of the series, if it converges.

$$\text{a) } \sum_{n=1}^{\infty} \frac{3^{n+1}}{4 \cdot 5^n} = \frac{9}{20} + \frac{27}{100} + \dots$$

$$= \sum_{n=1}^{\infty} \left(\frac{9}{20} \right) \left(\frac{3}{5} \right)^{n-1}$$

$$= \frac{\frac{9}{20}}{1 - \frac{3}{5}} \quad \text{geometric series with } |r| = \frac{3}{5} < 1$$

$$= \frac{9}{8}$$

$$r = \frac{\frac{27}{100}}{\frac{9}{20}} = \frac{27}{100} \cdot \frac{20}{9} = \frac{3}{5}$$

$$\text{b) } \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$$

$$s_n = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) + \dots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$$

$$= 1 - \frac{1}{\sqrt{n+1}}$$

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{\sqrt{n+1}} \right) = 1$$

$$\text{Ergo } \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = 1$$

Question 3 (12 points)

Determine if the series $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^2}$ converges absolutely, conditionally or diverges.

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n+3}{n^2} \right| = \sum_{n=1}^{\infty} \frac{n+3}{n^2} \quad \text{Let } b_n = \frac{n}{n^2} = \frac{1}{n}$$

$$\text{then } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+3}{n^2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2 + 3n}{n^2} = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{n} \right) = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), then $\sum_{n=1}^{\infty} \frac{n+3}{n^2}$ diverges by the limit comparison test.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^2} \quad b_n = \frac{n+3}{n^2}$$

$$(i) f(x) = \frac{x+3}{x^2}$$

$$f'(x) = \frac{x^2 - (x+3)2x}{x^4} = \frac{-x^2 - 6x}{x^4} = \frac{-x-6}{x^3} < 0 \quad \text{for } x \geq 1$$

Hence b_n is decreasing.

$$(ii) \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n+3}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{3}{n^2} \right) = 0$$

Thus $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^2}$ converges by the alternating series test.

Ergo $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^2}$ converges conditionally.

Question 4 (9 points)

Determine whether the series $\sum_{n=3}^{\infty} \frac{3}{n(\ln n)^4}$ is convergent or divergent.

$$\text{Let } f(x) = \frac{3}{x(\ln x)^4}$$

1. $f(x) \geq 0$ on $[3, \infty)$
2. $f'(x) = \frac{-3((\ln x)^4 + x4(\ln x)^3 \frac{1}{x})}{x^2(\ln x)^8}$
 $= \frac{-3(\ln x)^3(\ln x + 4)}{x^2(\ln x)^8}$
 $= \frac{-3(\ln x + 4)}{x^2(\ln x)^5} < 0$ for $x \geq 3$

thus f is decreasing on $[3, \infty)$.

3. f is continuous on $[3, \infty)$ (reciprocal of the product of two continuous functions, each continuous on their domain, hence f is continuous on its domain $(0, 1), (1, \infty)$)

$$\begin{aligned} \int_3^{\infty} \frac{3}{x(\ln x)^4} dx &= \lim_{t \rightarrow \infty} \int_3^t \frac{3}{x(\ln x)^4} dx \\ &= \lim_{t \rightarrow \infty} \int_{x=3}^{x=t} \frac{3}{u^4} du && \begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array} \\ &= \lim_{t \rightarrow \infty} \left[\frac{-1}{u^3} \right]_{x=3}^{x=t} \\ &= \lim_{t \rightarrow \infty} \left[\frac{-1}{(\ln x)^3} \right]_3^t \\ &= \lim_{t \rightarrow \infty} \left(\frac{-1}{(\ln t)^3} + \frac{1}{(\ln 3)^3} \right) \\ &= \frac{1}{(\ln 3)^3} \end{aligned}$$

Thus $\sum_{n=3}^{\infty} \frac{3}{x(\ln x)^4}$ converges by the integral test.

Question 5 (13 points)

Find the radius and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n n^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-1)^{n+1}}{2^{n+1}(n+1)^2}}{\frac{(x-1)^n}{2^n n^2}} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^n (x-1)^{n+1} n^2}{2^{n+1} (x-1)^n (n^2 + 2n + 1)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x-1}{2} \right| \frac{n^2}{n^2 + 2n + 1} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \left| \frac{x-1}{2} \right| \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}} \\ &= \left| \frac{x-1}{2} \right| \end{aligned}$$

Thus the power series converges if $\left| \frac{x-1}{2} \right| < 1$ by the ratio test.

$$\begin{aligned} \left| \frac{x-1}{2} \right| &< 1 \\ |x-1| &< 2 \quad \therefore R = 2 \\ -2 &< x-1 < 2 \\ -1 &< x < 3 \end{aligned}$$

$$\text{At } x = -1, \sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (} p\text{-series with } p = 2 > 1\text{), thus } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges absolutely, hence converges.

$$\text{At } x = 3, \sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (} p\text{-series with } p = 2 > 1\text{)}$$

Hence $I = [-1, 3]$ and $R = 2$.