

MATHEMATICS 201-203-RE

Integral Calculus

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Review Exercises

1. Evaluate using the definition of the definite integral as a Riemann Sum. Does the answer represent an area?

a) $\int_{-5}^2 (3x - 5) dx$

b) $\int_{-6}^{-1} (2x^2 - x - 1) dx$

c) $\int_{-1}^7 (6 + x)(1 - x) dx$

d) $\int_{-1}^4 (x^3 - 1) dx$

2. Find $f'(x)$ using the Fundamental Theorem of Calculus.

a) $f(x) = \int_4^{x^2} \sin t^2 dt$

b) $f(x) = \int_{2x}^{3x} \frac{1}{\ln t} dt$

3. Evaluate $\frac{d}{dx} \int_3^{8x} \frac{\sin t}{t} dt + \int_0^2 \frac{d}{dx} [\sqrt{1+x^3}] dx + \frac{d}{dx} \int_{-2}^7 e^{x^2+x} dx$

4. Evaluate the integral.

a) $\int \left(\frac{6}{x^8} + 4\sqrt[3]{x^2} - \frac{1}{5\sqrt{x}} \right) dx$

b) $\int \frac{x^2 - 5\sqrt{x} + 1}{x^3} dx$

c) $\int \sqrt[4]{x} (x^2 + 2\sqrt{x}) dx$

d) $\int (7^{2x-1} + (2x-1)^7) dx$

e) $\int \frac{\sin x}{1 + \cos^2 x} dx$

f) $\int \sin^3(5x) \cos^4(5x) dx$

g) $\int \frac{x^4 + x^2 + 1}{x^5 + 4x^3} dx$

h) $\int \frac{2x+5}{x^2 - 6x + 8} dx$

i) $\int \frac{1}{x+x^3} dx$

j) $\int \frac{2x^3 - 8x^2 + 9x + 1}{(x-2)^2} dx$

k) $\int \tan^3 x \sec^4 x dx$

l) $\int \cos \sqrt{x} dx$

m) $\int \arcsin x dx$

n) $\int e^{3x} \cos 2x dx$

o) $\int x^2 e^{-3x} dx$

p) $\int \sqrt{e^x - 1} dx$

q) $\int \frac{x+1}{(x^2 + 2x + 10)^{\frac{3}{2}}} dx$

r) $\int \frac{\ln(\ln x)}{x} dx$

s) $\int \frac{\sqrt[3]{x} + 1}{\sqrt[3]{x} - 1} dx$

t) $\int \frac{\sec x \tan x}{\sqrt{3 + 4 \sec x}} dx$

u) $\int \frac{2x-5}{x^2 + 6x + 34} dx$

v) $\int \frac{3x^3 + x^2 + 9x + 1}{(x^2 + 1)^2} dx$

w) $\int \frac{x^4 + 5}{x^3 + 3x^2 - 10x} dx$

x) $\int \csc^4 x \sqrt{\cot^5 x} dx$

y) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^4 x dx$

z) $\int_2^6 \frac{3x+4}{x^2 - 4x + 20} dx$

aa) $\int_1^e \frac{\ln x}{x^5} dx$

5. It is estimated that t months from now the population of a certain town will be changing at a rate of $\frac{dP}{dt} = 2 + t\sqrt{t+1}$ people per month. The current population is 5000. What will be the population 9 months from now?
6. After t hours on the job, a factory worker can produce units at a rate of $\frac{dQ}{dt} = t \ln(10-t)$ units per hour. How many units does a worker who arrives at 8:00 A.M. produce during the first 3 hours?
7. Find the average value of each function over the given interval.
- $f(x) = x \ln x$ $[1, e]$
 - $f(x) = \frac{x}{\sqrt{x+1}}$ $[8, 24]$
 - $f(x) = \frac{5}{x^2 - 4x + 3}$ $[4, 6]$
8. Suppose the number of items a new worker on an assembly line produces daily after t days on the job is given by $I(t) = 35 \ln(t+1)$. Find the average number of items produced daily by this employee after 10 days.
9. Determine whether the following improper integral converges or diverges.
- $\int_7^{\infty} \frac{1}{(x+2)^{\frac{5}{2}}} dx$
 - $\int_{-\infty}^3 \frac{x+1}{x^2+1} dx$
 - $\int_{-2}^{\infty} \frac{x}{(x-1)^2} dx$
 - $\int_1^6 \frac{3x}{\sqrt[3]{x^2-9}} dx$
10. Estimate $\int_1^9 \ln(1+x) dx$ to 6 decimal places with $n = 4$ using
- right endpoint approximation
 - left endpoint approximation
 - midpoint rule
 - trapezoidal rule
 - Simpson's rule
 - Find the maximum error of estimate for the midpoint rule.
 - Find the maximum error of estimate for the trapezoidal rule.
 - Find the maximum error of estimate for the Simpson's rule.
 - Find the exact value of $\int_1^9 \ln(1+x) dx$.

11. Find the area of the region R if
- R is bounded by the curves $y = (x - 1)^2$ and $y = 1 - x^2$.
 - R is bounded by the curves $y = x^2 - 4x$, the x -axis, $x = -2$ and $x = 2$.
 - R is bounded by the curves $x = 3 - y^2$ and $x - y = 1$.
 - R is bounded by the curve $y = x^3 - x$ and $y = 3x$.
12. Consider the region R bounded by $y = e^x$, $y = e^{-x}$ and $x = 2$.
- Find the volume of the solid obtained by rotating the region about the x -axis. Sketch the region and a typical disk, washer of shell.
 - Find the volume of the solid obtained by rotating the region about the line $x = 2$. Sketch the region and a typical disk, washer of shell.
 - Find the volume of the solid obtained by rotating the region about the line $y = 9$. Sketch the region and a typical disk, washer of shell.
 - Find the volume of the solid whose cross-sections perpendicular to the x -axis are squares.
 - Find the volume of the solid whose cross-sections perpendicular to the x -axis are equilateral triangles.
13. Consider the region R in the first quadrant bounded by the curves $y = x^3$, $y = 2x - x^2$.
- Sketch the region R and find the area.
 - Find the volume obtained by rotating R about the x -axis. Sketch the region and a typical disk, washer of shell.
 - Find the volume obtained by rotating R about the y -axis. Sketch the region and a typical disk, washer of shell.
14. Consider the region R bounded by $y = 6x - x^2$ and $y = x$. Find the volume of the solid if cross-sections perpendicular to the x -axis are
- Find the volume of the solid obtained by rotating the region about the line $x = 6$. Sketch the region and a typical disk, washer of shell.
 - Find the volume of the solid obtained by rotating the region about the line $y = 9$. Sketch the region and a typical disk, washer of shell.
 - Find the volume of the solid whose base is R and cross-sections perpendicular to the x -axis are squares
 - Find the volume of the solid whose base is R and cross-sections perpendicular to the x -axis are semi-circles.
 - Find the volume of the solid whose base is R and cross-sections perpendicular to the x -axis are isosceles right triangles with the hypotenuse as the base.

15. Consider the region R bounded by $y = 5 - x^2$ and $y = -4x$.
- Find the volume obtained by rotating R about the line $y = 5$. Sketch the region and a typical disk, washer or shell.
 - Find the volume obtained by rotating R about the line $x = -2$. Sketch the region and a typical disk, washer or shell.
 - Find the volume of the solid whose base is R and if cross-sections perpendicular to the x -axis are quarter circles, whose radius is on the base.
16. Consider the region R bounded by $x = 3y - y^2$ and $y = -x$.
- Find the volume obtained by rotating R about the line $y = 5$. Sketch the region and a typical disk, washer or shell.
 - Find the volume obtained by rotating R about the line $x = 4$. Sketch the region and a typical disk, washer or shell.
17. Find the volume of a frustum of a pyramid with square base of side b , square top of side a , and height h .
18. Find the length of the curve given by $y = \frac{2}{3}(x^2 - 1)^{\frac{3}{2}}$ between $x = 1$ and $x = 3$.
19. Find the length of the curve given by $x = \frac{1}{5}y^5 + \frac{1}{12y^3}$ between $(\frac{17}{60}, 1)$ and $(\frac{163}{480}, \frac{1}{2})$.
20. Find the arc length function for the curve $y = \ln(x^2 - 1)$ starting at the point $P_0(2, \ln(3))$.
21. The demand function for an Iphone is $p = D(x) = \frac{1000}{x+1}$ and the supply function is $p = S(x) = (x+1)^2$ where x is the number of units (in thousands) and p the price per unit (in \$). Find the consumers' and producers' surplus at the equilibrium point.
22. A company estimates that the revenue produced by a machine at time t will be $f(t) = 50 + 7t$ thousand dollars per year. Find the total revenue, the present value and the future value of the machine over the next 5 years, if money is worth 5% compounded continuously. Also, find the capital value.
23. Find the *Gini* index for the Lorenz curve $L(x) = \frac{4}{\pi}x \arctan(x)$.
24. Solve the following differential equations.
- $\frac{dy}{dx} = \frac{(x^2 + 1)(y^2 + 1)}{x^2 - 1}$
 - $\frac{dy}{dx} = \sec y \arctan x$
 - $x^2 y dx + e^x \ln y dy = 0$
 - $\frac{dy}{dx} = y \sin^2 x \quad y(\frac{\pi}{2}) = 2$

25. If a new product becomes known both by word of mouth and by advertising, then p , the proportion of people who have heard of the new product after t weeks, satisfies a differential equation and initial condition of the form

$$\frac{dp}{dt} = 0.1(1-p) + 0.1p(1-p)$$

If initially 30% of the population have heard of the new product, what will be the proportion of have heard of it t weeks from now?

26. Determine if the following sequences are monotonic, bounded and convergent.

a) $a_n = \frac{\sqrt{n}}{3^n + 1}$

b) $\left\langle 5 - \left(-\frac{2}{3}\right)^n \right\rangle$

c) $\left\langle \frac{(-1)^n}{n!} \right\rangle$

d) $a_n = \frac{n}{\ln(n+1)}$

27. Determine if the following series converge. If so, find the sum.

a) $\sum_{n=1}^{\infty} \frac{4^{2n+1}}{3^{3n-1}}$

b) $\frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \dots$

c) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{n!}$

e) $\frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \dots$

f) $\sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$

28. Determine whether the following series are convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{2\sqrt{n} + 1}{n(1+n^2)}$

b) $\sum_{n=1}^{\infty} \frac{2^{2n} + 1}{5^n - 3}$

c) $\sum_{n=2}^{\infty} \frac{e^{\sqrt{n}}}{4\sqrt{n}}$

d) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$

e) $\sum_{n=1}^{\infty} \frac{n^3}{3^{n+1}}$

f) $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)!}$

g) $\sum_{n=1}^{\infty} \left(\frac{3}{3+5n}\right)^{2n}$

h) $\sum_{n=0}^{\infty} \arctan n$

29. Determine whether the series converges absolutely, converges conditionally or diverges.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3 + 4}}$

30. Find the radius and interval of convergence for the following power series.

a) $\sum_{n=1}^{\infty} \frac{(n-1)x^n}{n2^n}$

b) $\sum_{n=1}^{\infty} \frac{(3x-4)^n}{5^n}$

c) $\sum_{n=1}^{\infty} \frac{(7-2x)^n}{n(n+1)}$

d) $\sum_{n=1}^{\infty} \frac{(x+4)^n}{\sqrt{n}}$

e) $\sum_{n=1}^{\infty} \frac{5^n (2x-1)^n}{3 \cdot 10 \cdot 17 \cdots (7n-4)}$

f) $\sum_{n=1}^{\infty} \frac{(3n)!(x+\frac{1}{2})^n}{3^n (2n)!}$

31. Find the Maclaurin Series for $f(x)$ using the definition.

a) $f(x) = \frac{1}{1+2x}$ b) $f(x) = \cos(4x)$ c) $\sqrt[3]{x+8}$

32. Find the Taylor series for $f(x)$ at $x = a$.

a) $f(x) = \sin 3x$ $a = \frac{\pi}{3}$ b) $f(x) = \frac{1}{(6-5x)^2}$ $a = 1$

c) $f(x) = \sqrt[3]{x}$ $a = 32$

33. Use the known Maclaurin series for e^x , $\sin x$, $\cos x$ or $\frac{1}{1-x}$ to obtain the Maclaurin series for the given function. Give the radius of convergence.

a) $f(x) = \sin 3x$ b) $f(x) = \frac{1 - \cos 2x}{x}$ c) $f(x) = \frac{x}{1+x^2}$

34. Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

a) $f(x) = \sin x \cos x$ b) $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

35. Approximate f by a Taylor polynomial with degree n at the number a . Use this approximation to estimate the given number, and estimate the accuracy of that approximation.

a) $f(x) = \sqrt{1+2x}$ $n = 3$ $a = 4$ $\sqrt{9.2}$

b) $f(x) = \sec x$ $n = 2$ $a = \frac{\pi}{4}$ $\sec(43^\circ)$

36. Using the Maclaurin series for e^x , $\sin x$, $\cos x$ or $\frac{1}{1-x}$ to approximate the definite integral to within five decimal places.

a) $\int_0^{\frac{1}{2}} x \sin(x^3) dx$ b) $\int_0^{\frac{1}{5}} e^{\frac{-x^2}{2}} dx$ c) $\int_0^{\frac{1}{3}} \frac{1}{1+x^4} dx$

37. Consider the function $f(x) = \ln(2x+1)$.

a) Find the Maclaurin series for $f(x)$ using the definition.

b) Use differentiation, as well as your answer in (a) to find the Maclaurin series for $\frac{2}{2x+1}$

c) Use the answer from (a) to approximate the definite integral $\int_0^{\frac{1}{10}} \ln(1+2x) dx$ to within five decimal places.

Answers

1. a) $\frac{-133}{2}$ No b) $\frac{935}{6}$ Yes c) $\frac{-560}{3}$ No d) $\frac{235}{4}$ No
2. a) $2x \sin(x^4)$ b) $\frac{3}{\ln 3x} - \frac{2}{\ln 2x}$ 3. $\frac{\sin 8x}{x} + 2$
4. a) $\frac{-6}{7x^7} + \frac{12}{5}x^{\frac{5}{3}} - \frac{6}{25}x^{\frac{5}{6}} + C$ b) $\ln|x| + \frac{10}{3x^{\frac{3}{2}}} - \frac{1}{2x^2} + C$
- c) $\frac{4}{13}x^{\frac{13}{4}} + \frac{8}{7}x^{\frac{7}{4}} + C$ d) $\frac{7^{2x-1}}{2 \ln 7} + \frac{1}{16}(2x-1)^8 + C$
- e) $-\arctan(\cos x) + C$ f) $-\frac{1}{25}\cos^5 5x + \frac{1}{35}\cos^7 5x + C$
- g) $-\frac{1}{8x^2} + \frac{3}{16}\ln|x| + \frac{13}{32}\ln|x^2 + 4| + C$ h) $\frac{13}{2}\ln|x-4| - \frac{9}{2}\ln|x-2| + C$
- i) $\ln|x| - \frac{1}{2}\ln(x^2 + 1) + C$ j) $x^2 - \frac{3}{x-2} + \ln|x-2| + C$
- k) $\frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$ l) $2\cos\sqrt{x} + 2\sqrt{x}\sin\sqrt{x} + C$
- m) $x \arcsin x + \sqrt{1-x^2} + C$ n) $\frac{3}{13}e^{3x}\cos 2x + \frac{2}{13}e^{3x}\sin 2x + C$
- o) $-\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$ p) $2\sqrt{e^x - 1} - 2\arctan\sqrt{e^x - 1} + C$
- q) $\frac{-1}{\sqrt{x^2 + 2x + 10}} + C$ r) $\ln x \ln(\ln x) - \ln x + C$
- s) $x + 3x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 6\ln|x^{\frac{1}{3}} - 1| + C$ t) $\frac{1}{2}\sqrt{3 + 4\sec x} + C$
- u) $\ln|x^2 + 6x + 34| - \frac{11}{5}\arctan\frac{x+3}{5} + C$ v) $\frac{3}{2}\ln|x^2 + 1| + \arctan x - \frac{3}{x^2 + 1} + C$
- w) $\frac{1}{2}x^2 - 3x + 18\ln|x+5| - \frac{1}{2}\ln|x| + \frac{3}{2}\ln|x-2| + C$
- x) $\frac{-2}{7}\cot^{\frac{7}{2}}x - \frac{2}{11}\cot^{\frac{11}{2}}x + C$ y) $\frac{1}{4} + \frac{7\pi}{32} + \frac{7\sqrt{3}}{64}$
- z) $\frac{3}{2}\ln 2 + \frac{5\pi}{8}$ aa) $\frac{1}{16} - \frac{5}{16e^4}$
5. 5124
6. 9 units
7. $\frac{e^2+1}{4e-4}$ b) $\frac{23}{6}$ c) $\frac{5}{2}\ln 3 - \frac{5}{4}\ln 5$
8. 57.3 items
9. a) $\frac{2}{81}$ b) Div c) Div d) $\frac{45}{4}$
10. a) 15.120161 b) 11.901285 c) 13.702370 d) 13.510723
- e) 13.628130 f) $\frac{1}{3}$ g) $\frac{2}{3}$ h) $\frac{4}{15}$
- i) $10\ln 10 - 2\ln 2 - 8 \approx 13.639557$
11. a) $\frac{1}{3}$ b) 16 c) $\frac{9}{2}$ d) 8
12. a) $\pi\left(\frac{e^4}{2} + \frac{1}{2e^4} - 1\right)$ b) $2\pi\left(e^2 - \frac{1}{e^2} - 4\right)$ c) $\pi\left(\frac{18}{e^2} - \frac{1}{2e^4} + 18e^2 - \frac{1}{2}e^4 - 35\right)$
- d) $\frac{1}{2}e^4 - 4 - \frac{1}{2e^4}$ e) $\frac{\sqrt{3}}{8}e^4 - \sqrt{3} - \frac{\sqrt{3}}{8e^4}$
13. a) $\frac{5}{12}$ b) $\frac{41\pi}{105}$ c) $\frac{13\pi}{30}$

14. a) $\frac{875\pi}{6}$ b) $\frac{500\pi}{3}$ c) $\frac{625}{6}$ d) $\frac{625\pi}{48}$ e) $\frac{625}{24}$
15. a) $\frac{3384\pi}{5}$ b) 288π c) $\frac{324\pi}{5}$
16. a) 64π b) $\frac{1408\pi}{15}$
17. $\frac{1}{3}h(a^2 + ab + b^2)$ 18. $\frac{46}{3}$ 19. $\frac{373}{480}$
20. $s(x) = x + \ln|x-1| - \ln|x+1| - 2 + \ln 3$
21. CS = \$1 402 585 PS = \$567 000
22. TR = \$337 500 P = \$295 396 A = \$379 297 CV = \$3 800 000
23. $\frac{4-\pi}{\pi} \approx 0.273$
24. a) $\arctan y = x + \ln|x-1| - \ln|x+1| + C$ b) $\sin y = x \arctan x - \frac{1}{2} \ln(x^2 + 1) + C$
 c) $(\ln y)^2 = 2x^2 e^{-x} + 4x e^{-x} + 4e^{-x} + C$ d) $y = 2e^{\frac{1}{2}x - \frac{1}{4} \sin 2x - \frac{\pi}{4}}$
25. $p(t) = \frac{13e^{\frac{1}{3}t} - 7}{13e^{\frac{1}{3}t} + 7}$
26. a) decreasing, $0 \leq a_n \leq \frac{1}{4}$, conv to 0 b) nonmonotonic, $\frac{41}{9} \leq a_n \leq \frac{17}{3}$, conv to 5
 c) nonmonotonic, $-1 \leq a_n \leq \frac{1}{2}$, conv to 0 d) increasing, $0 \leq a_n$, not bounded above, div
27. a) $\frac{192}{11}$ b) $\frac{1}{5}$ c) $\frac{3}{2}$ d) $\frac{3}{e^3}$ e) 1 f) Diverges
28. a) Converges b) Converges c) Diverges d) Converges
 e) Converges f) Converges g) Converges h) Diverges
29. a) Converges conditionally b) Converges conditionally
30. a) $R = 2$, $I = (-2, 2)$ b) $R = \frac{5}{3}$, $I = (-\frac{1}{3}, 3)$
 c) $R = \frac{1}{2}$, $I = [3, 4]$ d) $R = 1$, $I = [-5, -3]$
 e) $R = \infty$, $I = \mathbb{R}$ f) $R = 0$, $I = \{\frac{-1}{2}\}$
31. a) $\sum_{n=0}^{\infty} (-1)^n 2^n x^n$ b) $\sum_{n=0}^{\infty} (-1)^n \frac{16^n x^{2n}}{(2n)!}$ c) $2 + \frac{1}{12}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2 \cdot 5 \cdot 8 \cdots (3n-4) x^n}{3^n 2^{3n-1} n!}$
32. a) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n-1} (x - \frac{\pi}{3})^{2n-1}}{(2n-1)!}$ b) $\sum_{n=0}^{\infty} (n+1) 5^n (x-1)^n$
 c) $2 + \frac{1}{80}(x-32) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)(x-32)^n}{5^n 2^{5n-1} n!}$
33. a) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$, $R = \infty$ b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n} x^{2n-1}}{(2n)!}$, $R = \infty$ c) $\sum_{n=0}^{\infty} (-1)^n x^{2n+1}$, $R = 1$
34. a) $x - \frac{2}{3}x + \frac{2}{15}x^5 - \dots$ b) $x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \dots$
35. a) $\sqrt{1+2x} \approx 3 + \frac{1}{3}(x-4) - \frac{1}{54}(x-4)^2 + \frac{1}{486}(x-4)^3$ $\sqrt{9.2} \approx 3.0331502$ $|R_3(x)| < 0.000000031$
 b) $\sec x \approx \sqrt{2} + \sqrt{2}(x - \frac{\pi}{4}) + \frac{3\sqrt{2}}{2}(x - \frac{\pi}{4})^2$ $\sec(43^\circ) \approx 1.36743$ $|R_3(x)| < 0.00013$
36. a) 0.00624 b) 0.19867 c) 0.33252
37. a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n} x^n$ b) $\sum_{n=0}^{\infty} (-1)^n 2^{n+1} x^n$ c) 0.00939