

MATHEMATICS 201-203-RE

Integral Calculus

Martin Huard

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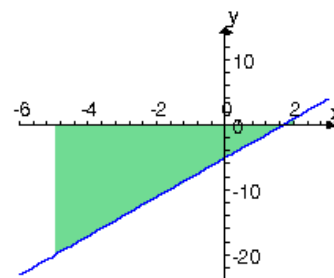
Review Exercises SOLUTIONS

1. Evaluate using the definition of the definite integral as a Riemann Sum. Does the answer represent an area?

$$\begin{aligned} \text{a) } \int_{-5}^2 (3x-5) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[3 \left(-5 + \frac{7i}{n} \right) - 5 \right] \frac{7}{n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{147}{n^2} \sum_{i=1}^n i - \frac{140}{n} \sum_{i=1}^n 1 \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{147}{n^2} \frac{n(n+1)}{2} - \frac{140}{n} n \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{147}{2} + \frac{147}{2n} - 140 \right) \\ &= \frac{-133}{2} \end{aligned}$$

$$\Delta x = \frac{b-a}{n} = \frac{2-(-5)}{n} = \frac{7}{n}$$

$$x_i = a + i\Delta x = -2 + \frac{7i}{n}$$

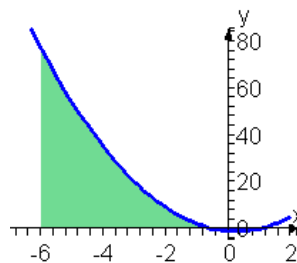


No, the answer represents the negative of the area.

$$\begin{aligned} \text{b) } \int_{-6}^{-1} (2x^2 - x - 1) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[2 \left(-6 + \frac{5i}{n} \right)^2 - \left(-6 + \frac{5i}{n} \right) - 1 \right] \frac{5}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{250i^2}{n^3} - \frac{625i}{n^2} + \frac{385}{n} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{250}{n^3} \sum_{i=1}^n i^2 - \frac{625}{n^2} \sum_{i=1}^n i + \frac{385}{n} \sum_{i=1}^n 1 \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{250}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{625}{n^2} \frac{n(n+1)}{2} + \frac{385}{n} n \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{250}{3} + \frac{125}{n} + \frac{125}{3n^2} - \frac{625}{2} - \frac{325}{2n} + 385 \right) \\ &= \frac{935}{6} \end{aligned}$$

$$\Delta x = \frac{b-a}{n} = \frac{-1+6}{n} = \frac{5}{n}$$

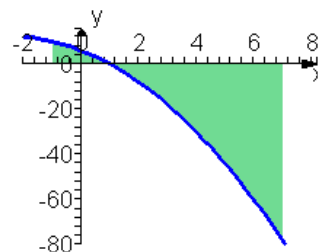
$$x_i = a + i\Delta x = -6 + \frac{5i}{n}$$



Yes since the function $f(x) = 2x^2 - x - 1$ is positive on the interval $[-6, -1]$

$$\begin{aligned}
 \text{c) } \int_{-1}^7 (6+x)(1-x) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(6 - 1 + \frac{8i}{n} \right) \left(1 + 1 - \frac{8i}{n} \right) \right] \frac{8}{n} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{-512i^2}{n^3} - \frac{192i}{n^2} + \frac{80}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{-512}{n^3} \sum_{i=1}^n i^2 - \frac{192}{n^2} \sum_{i=1}^n i + \frac{80}{n} \sum_{i=1}^n 1 \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{-512}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{192}{n^2} \frac{n(n+1)}{2} + \frac{80}{n} n \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{-512}{3} - \frac{256}{n} - \frac{256}{3n^2} - 96 - \frac{96}{n} + 80 \right) \\
 &= -\frac{560}{3}
 \end{aligned}$$

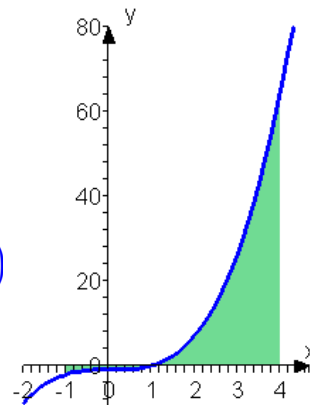
$$\begin{aligned}
 \Delta x &= \frac{b-a}{n} = \frac{7+1}{n} = \frac{8}{n} \\
 x_i &= a + i\Delta x = -1 + \frac{8i}{n}
 \end{aligned}$$



No, the answer represents the difference in the area above and below the x -axis.

$$\begin{aligned}
 \text{d) } \int_{-1}^4 (x^3 - 1) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(-1 + \frac{5i}{n} \right)^3 - 1 \right] \frac{5}{n} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{625i^3}{n^4} - \frac{375i^2}{n^3} + \frac{75i}{n^2} - \frac{10}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{625}{n^4} \sum_{i=1}^n i^3 - \frac{375}{n^3} \sum_{i=1}^n i^2 + \frac{75}{n^2} \sum_{i=1}^n i - \frac{10}{n} \sum_{i=1}^n 1 \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{625}{n^4} \frac{n^2(n+1)^2}{4} - \frac{375}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{75}{n^2} \frac{n(n+1)}{2} - \frac{10}{n} n \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{625}{4} + \frac{625}{2n} + \frac{625}{4n^2} - 125 - \frac{375}{2n} - \frac{125}{2n^2} + \frac{75}{2} + \frac{75}{n} - 10 \right) \\
 &= \frac{235}{4}
 \end{aligned}$$

$$\begin{aligned}
 \Delta x &= \frac{b-a}{n} = \frac{4+1}{n} = \frac{5}{n} \\
 x_i &= a + i\Delta x = -1 + \frac{5i}{n}
 \end{aligned}$$



No, the answer represents the difference in the area above and below the x -axis.

2. Find $f'(x)$ using the Fundamental Theorem of Calculus.

$$\text{a) } \int_4^{x^2} \sin t^2 dt$$

$$\begin{aligned}
 f(x) &= \int_4^{x^2} \sin(t^2) dt & f'(x) &= \frac{d}{dx} \left[\int_4^{x^2} \sin(t^2) dt \right] \\
 & & &= \frac{d}{du} \left[\int_4^u \sin(t^2) dt \right] \frac{du}{dx} & u &= x^2 \\
 & & &= \sin(u^2) 2x \\
 & & &= 2x \sin(x^4)
 \end{aligned}$$

$$\text{b) } \int_{2x}^{3x} \frac{1}{\ln t} dt$$

$$\begin{aligned} f(x) &= \int_{2x}^{3x} \frac{1}{\ln t} dt \\ &= \int_{2x}^1 \frac{1}{\ln t} dt + \int_1^{3x} \frac{1}{\ln t} dt \\ &= -\int_1^{2x} \frac{1}{\ln t} dt + \int_1^{3x} \frac{1}{\ln t} dt \\ f'(x) &= -\frac{d}{dx} \left[\int_1^{2x} \frac{1}{\ln t} dt \right] + \frac{d}{dx} \left[\int_1^{3x} \frac{1}{\ln t} dt \right] & u = 2x \\ &= -\frac{d}{du} \left[\int_0^u \frac{1}{\ln t} dt \right] \frac{du}{dx} + \frac{d}{dv} \left[\int_0^v \frac{1}{\ln t} dt \right] \frac{dv}{dx} & v = 3x \\ &= -\frac{1}{\ln u} 2 + \frac{1}{\ln v} 3 \\ &= \frac{3}{\ln 3x} - \frac{2}{\ln 2x} \end{aligned}$$

$$\begin{aligned} 3. \text{ Evaluate } & \frac{d}{dx} \int_3^{8x} \frac{\sin t}{t} dt + \int_0^2 \frac{d}{dx} \left[\sqrt{1+x^3} \right] dx + \frac{d}{dx} \int_{-2}^7 e^{x^2+x} dx \\ & \frac{d}{dx} \int_3^{8x} \frac{\sin t}{t} dt + \int_0^2 \frac{d}{dx} \left[\sqrt{1+x^3} \right] dx + \frac{d}{dx} \int_{-2}^7 e^{x^2+x} dx = \frac{d}{du} \left[\int_3^u \frac{\sin t}{t} dt \right] \frac{du}{dx} + \left[\sqrt{1+x^3} \right]_0^2 + 0 \\ & = \frac{\sin u}{u} (8) + \sqrt{9} - \sqrt{1} \\ & = \frac{8 \sin 8x}{8x} + 2 \\ & = \frac{\sin 8x}{x} + 2 \end{aligned}$$

4. Evaluate the integral.

$$\begin{aligned} \text{a) } \int \left(\frac{6}{x^8} + 4\sqrt[3]{x^2} - \frac{1}{5\sqrt{x}} \right) dx &= \int \left(6x^{-8} + 4x^{\frac{2}{3}} - \frac{1}{5}x^{-\frac{1}{2}} \right) dx \\ &= \frac{-6}{7}x^{-7} + 4\left(\frac{3}{5}\right)x^{\frac{5}{3}} - \frac{1}{5}\frac{6}{5}x^{\frac{5}{2}} + C \\ &= \frac{-6}{7x^7} + \frac{12}{5}x^{\frac{5}{3}} - \frac{6}{25}x^{\frac{5}{2}} + C \end{aligned}$$

$$\begin{aligned} \text{b) } \int \frac{x^2 - 5\sqrt{x} + 1}{x^3} dx &= \int \left(\frac{x^2}{x^3} - \frac{5x^{\frac{1}{2}}}{x^3} + \frac{1}{x^3} \right) dx \\ &= \int \left(x^{-1} - 5x^{-\frac{5}{2}} + x^{-3} \right) dx \\ &= \ln|x| - 5\left(\frac{-2}{3}\right)x^{-\frac{3}{2}} + \left(\frac{-1}{2}\right)x^{-2} + C \\ &= \ln|x| + \frac{10}{3x^{\frac{3}{2}}} - \frac{1}{2x^2} + C \end{aligned}$$

$$\begin{aligned} \text{c) } \int \sqrt[4]{x}(x^2 + 2\sqrt{x}) dx &= \int (x^{\frac{9}{4}} + 2x^{\frac{3}{4}}) dx \\ &= \frac{4}{13} x^{\frac{13}{4}} + \frac{8}{7} x^{\frac{7}{4}} + C \end{aligned}$$

$$\text{d) } \int (7^{2x-1} + (2x-1)^7) dx = \frac{7^{2x-1}}{2 \ln 7} + \frac{1}{16} (2x-1)^8 + C$$

$$\begin{aligned} \text{e) } \int \frac{\sin x}{1 + \cos^2 x} dx &= \int \frac{-1}{1+u^2} du && u = \cos x \\ &= -\arctan u + C && du = -\sin x dx \\ &= -\arctan(\cos x) + C \end{aligned}$$

$$\begin{aligned} \text{f) } \int \sin^3(5x) \cos^4(5x) dx &= \int \sin(5x)(1 - \cos^2(5x)) \cos^4(5x) dx && u = \cos 5x \\ &= \int -\frac{1}{5}(1-u^2)u^4 du && du = -5 \sin 5x dx \\ &= \int (\frac{1}{5}u^6 - \frac{1}{5}u^4) du \\ &= \frac{1}{35}u^7 - \frac{1}{25}u^5 + C \\ &= \frac{1}{35} \cos^7(5x) - \frac{1}{25} \cos^5(5x) + C \end{aligned}$$

$$\text{g) } \int \frac{x^4 + x^2 + 1}{x^5 + 4x^3} dx$$

$$\begin{aligned} \frac{x^4 + x^2 + 1}{x^3(x^2 + 4)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + 4} \\ &= \frac{Ax^2(x^2 + 4) + Bx(x^2 + 4)x + C(x^2 + 4) + (Dx + E)x^3}{x^3(x^2 + 4)} \end{aligned}$$

$$\begin{aligned} \therefore x^4 + x^2 + 1 &= Ax^2(x^2 + 4) + Bx(x^2 + 4)x + C(x^2 + 4) + (Dx + E)x^3 \\ &= (A + D)x^4 + (B + E)x^3 + (4A + C)x^2 + 4Bx + 4C \end{aligned}$$

$$\text{Thus } x^4 : 1 = A + D \quad \Rightarrow D = \frac{13}{16}$$

$$x^3 : 0 = B + E \quad \Rightarrow E = 0$$

$$x^2 : 1 = 4A + C \quad \Rightarrow A = \frac{3}{16}$$

$$x : 0 = 4B \quad \Rightarrow B = 0$$

$$: 1 = 4C \quad \Rightarrow C = \frac{1}{4}$$

$$\begin{aligned} \int \frac{x^4 + x^2 + 1}{x^5 + 4x^3} dx &= \int \left(\frac{3}{16x} + \frac{1}{4x^3} + \frac{\frac{13}{16}x}{x^2 + 4} \right) dx \\ &= \frac{3}{16} \ln|x| - \frac{1}{8x^2} + \frac{13}{16} \int \frac{\frac{1}{2} du}{u} && u = x^2 + 4 \\ &= \frac{3}{16} \ln|x| - \frac{1}{8x^2} + \frac{13}{32} \ln|x^2 + 4| + C && du = 2x dx \\ & && \frac{1}{2} du = x dx \end{aligned}$$

$$\begin{aligned}
 \text{h) } \int \frac{2x+5}{x^2-6x+8} dx & \\
 \frac{2x+5}{x^2-6x+8} &= \frac{2x+5}{(x-4)(x-2)} = \frac{A}{x-4} + \frac{B}{x-2} \\
 &= \frac{A(x-2)+B(x-4)}{(x-4)(x-2)} \\
 \therefore 2x+5 &= (A+B)x - 2A - 4B \\
 2 &= A+B & 4 &= 2A+2B \\
 5 &= -2A-4B & 5 &= -2A-4B \\
 & & 9 &= -2B \\
 & & B &= \frac{-9}{2} & A &= \frac{13}{2} \\
 \int \frac{2x+5}{x^2-6x+8} dx &= \int \left(\frac{\frac{13}{2}}{x-4} - \frac{\frac{9}{2}}{x-2} \right) dx \\
 &= \frac{13}{2} \ln|x-4| - \frac{9}{2} \ln|x-2| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{i) } \int \frac{1}{x+x^3} dx & \\
 \frac{1}{x(1+x^2)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)} \\
 \therefore 1 &= A(x^2+1) + (Bx+C)x \\
 &= (A+B)x^2 + Cx + A \\
 \text{Thus } x^2: 0 &= A+B & \Rightarrow B &= -1 \\
 x: 0 &= C \\
 1 &= A & \Rightarrow A &= 1 \\
 \int \frac{1}{x+x^3} dx &= \int \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx & u &= x^2+1 \\
 &= \ln|x| - \int \frac{\frac{1}{2}}{u} du & du &= 2x dx \\
 &= \ln|x| - \frac{1}{2} \ln|u| + C & \frac{1}{2} du &= x dx \\
 &= \ln|x| - \frac{1}{2} \ln|x^2+1| + C
 \end{aligned}$$

$$j) \int \frac{2x^3 - 8x^2 + 9x + 1}{(x-2)^2} dx$$

$$\begin{array}{l} x^2 - 4x + 4 \overline{) 2x^3 - 8x^2 + 9x + 1} \\ \underline{2x^3 - 8x^2 + 8x} \\ x + 1 \end{array}$$

$$\frac{2x^3 - 8x^2 + 9x + 1}{(x-2)^2} = 2x + \frac{x+1}{(x-2)^2}$$

$$\frac{x+1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{A(x-2) + B}{(x-2)^2}$$

$$\therefore x+1 = Ax - 2A + B$$

$$\text{Thus } x: 1 = A$$

$$: 1 = -2A + B \Rightarrow B = 3$$

$$\int \frac{2x^3 - 8x^2 + 9x + 1}{(x-2)^2} dx = \int \left(2x + \frac{1}{x-2} + \frac{3}{(x-2)^2} \right) dx$$

$$= x^2 + \ln|x-2| - \frac{3}{x-2} + C$$

$$k) \int \tan^3 x \sec^4 x dx = \int \tan^3 x (1 + \tan^2 x) \sec^2 x dx$$

$$= \int u^3 (1 + u^2) du$$

$$= \int (u^3 + u^5) du$$

$$= \frac{1}{4}u^4 + \frac{1}{6}u^6 + C$$

$$= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C$$

$$u = \tan x$$

$$du = \sec^2 x dx$$

$$l) \int \cos \sqrt{x} dx = \int 2v \cos v dv$$

$$= 2v \sin v - \int 2 \sin v dv$$

$$= 2v \sin v + 2 \cos v + C$$

$$= 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$$

$$v = \sqrt{x}$$

$$v^2 = x$$

$$2v dv = dx$$

$$u = 2v$$

$$V = \sin v$$

$$du = 2 dx$$

$$dV = \cos v dv$$

$$m) \int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \arcsin x - \int \frac{\frac{1}{2}}{\sqrt{u}} du$$

$$= x \arcsin x + \sqrt{u} + C$$

$$= x \arcsin x + \sqrt{1-x^2} + C$$

$$u = \arcsin x$$

$$v = x$$

$$du = \frac{1}{\sqrt{1-x^2}} dx$$

$$dv = dx$$

$$u = 1 - x^2$$

$$du = -2x dx$$

$$\frac{-1}{2} du = x dx$$

$$\begin{aligned} \text{n) } \int e^{3x} \cos 2x \, dx &= \frac{1}{2} e^{3x} \sin 2x - \frac{3}{2} \int e^{3x} \sin 2x \, dx & u &= e^{3x} & v &= \frac{1}{2} \sin 2x \\ &= \frac{1}{2} e^{3x} \sin 2x + \frac{3}{4} e^{3x} \cos 2x - \frac{9}{4} \int e^{3x} \cos 2x \, dx & du &= 3e^{3x} dx & dv &= \cos 2x \, dx \\ & & u &= e^{3x} & v &= \frac{1}{2} \cos 2x \\ & & du &= 3e^{3x} dx & dv &= \sin 2x \, dx \end{aligned}$$

$$\frac{13}{4} \int e^{3x} \cos 2x \, dx = \frac{1}{2} e^{3x} \sin 2x + \frac{3}{4} e^{3x} \cos 2x + C$$

$$\int e^{3x} \cos 2x \, dx = \frac{2}{13} e^{3x} \sin 2x + \frac{3}{13} e^{3x} \cos 2x + C$$

$$\begin{aligned} \text{o) } \int x^2 e^{-3x} \, dx &= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} \, dx & u &= x^2 & v &= \frac{-1}{3} e^{-3x} \\ &= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} + \frac{2}{9} \int e^{-3x} \, dx & du &= 2x dx & dv &= e^{-3x} dx \\ &= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C & u &= x & v &= \frac{-1}{3} e^{-3x} \\ & & du &= dx & dv &= e^{-3x} dx \end{aligned}$$

$$\begin{aligned} \text{p) } \int \sqrt{e^x - 1} \, dx &= \int \frac{2v^2}{v^2 + 1} \, dv & v &= \sqrt{e^x - 1} \\ &= \int \left(2 - \frac{2}{v^2 + 1} \right) \, dv & v^2 &= e^x - 1 & v^2 + 1 &= \sqrt{2v^2} \\ &= 2v - 2 \arctan v + C & v^2 + 1 &= e^x & &= \frac{2v^2 + 2}{-2} \\ &= 2\sqrt{e^x - 1} - 2 \arctan \sqrt{e^x - 1} + C & \ln(v^2 + 1) &= x & & \\ & & \frac{2v}{v^2 + 1} \, dv &= dx & & \end{aligned}$$

$$\begin{aligned} \text{q) } \int \frac{x+1}{(x^2+2x+10)^{\frac{3}{2}}} \, dx &= \int \frac{1}{2u^{\frac{3}{2}}} \, dx & u &= x^2 + 2x + 10 \\ &= \int \frac{1}{2} u^{-\frac{3}{2}} \, dx & du &= (2x+2) \, dx \\ &= \frac{1}{2} (-2) u^{-\frac{1}{2}} + C & \frac{1}{2} du &= (x+1) \, dx \\ &= \frac{-1}{\sqrt{u}} + C \\ &= \frac{-1}{\sqrt{x^2+2x+10}} + C \end{aligned}$$

$$\begin{aligned} \text{r) } \int \frac{\ln(\ln x)}{x} \, dx &= \int \ln u \, du & u &= \ln x \\ &= u \ln u - \int du & du &= \frac{1}{x} dx \\ &= u \ln u - u + C & U &= \ln u & v &= u du \\ &= \ln x \ln(\ln x) - \ln x + C & dU &= \frac{1}{u} du & dv &= du \end{aligned}$$

$$\begin{aligned}
 \text{s) } \int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1} dx &= \int \frac{v+1}{v-1} 3v^2 dv & v &= \sqrt[3]{x} \\
 &= \int \frac{3v^3+3v^2}{v-1} dv & v^3 &= x \\
 &= \int \left(3v^2+6v+6+\frac{6}{v-1} \right) dv & 3v^2 dv &= dx \\
 &= v^3+3v^2+6v+6\ln|v-1|+C & & \\
 &= x+3x^{\frac{2}{3}}+6x^{\frac{1}{3}}+6\ln|\sqrt[3]{x}-1|+C & & \\
 & & & \frac{3v^2+6v+6}{3v^3+3v^2} \\
 & & & \frac{3v^3-3v^2}{6v^2} \\
 & & & \frac{6v^2-6v}{6v} \\
 & & & \frac{6v-6}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{t) } \int \frac{\sec x \tan x}{\sqrt{3+4\sec x}} dx &= \int \frac{1}{4\sqrt{u}} du & u &= 3+4\sec x \\
 &= \frac{1}{2}\sqrt{u}+C & du &= 4\sec x \tan x dx \\
 &= \frac{1}{2}\sqrt{3+4\sec x}+C & &
 \end{aligned}$$

$$\begin{aligned}
 \text{u) } \int \frac{2x-5}{x^2+6x+34} dx &= \int \frac{2x-5}{(x+3)^2+25} dx & x^2+6x+34 &= x^2+6x+\left(\frac{6}{2}\right)^2+34-\left(\frac{6}{2}\right)^2 \\
 &= \int \frac{2(u-3)-5}{u^2+25} du & &= (x+3)^2+25 \\
 &= \int \frac{2u-11}{u^2+25} du & u &= x+3 \\
 &= \int \frac{2u}{u^2+25} du - 11 \int \frac{1}{u^2+25} du & du &= dx \\
 &= \int \frac{1}{v} dv - \frac{11}{5} \arctan \frac{u}{5} + C & v &= u^2+25 \\
 &= \ln|v| - \frac{11}{5} \arctan \frac{u}{5} + C & dv &= 2u du \\
 &= \ln|u^2+25| - \frac{11}{5} \arctan \frac{u}{5} + C & \frac{1}{2} dv &= u du \\
 &= \ln|x^2+6x+34| - \frac{11}{5} \arctan \frac{x+3}{5} + C & &
 \end{aligned}$$

$$\begin{aligned}
 \text{v) } \int \frac{3x^3+x^2+9x+1}{(x^2+1)^2} dx & \\
 \frac{3x^3+x^2+9x+1}{(x^2+1)^2} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \\
 &= \frac{(Ax+B)(x^2+1)+Cx+D}{(x^2+1)^2} \\
 3x^3+x^2+9x+1 &= (Ax+B)(x^2+1)+Cx+D \\
 3x^3+x^2+9x+1 &= Ax^3+Bx^2+(A+C)x+B+D
 \end{aligned}$$

$$x^3 : 3 = A$$

$$x^2 : 1 = B$$

$$x : 9 = A + C \Rightarrow C = 6$$

$$cst : 1 = B + D \Rightarrow D = 0$$

$$\begin{aligned} \int \frac{3x^3 + x^2 + 9x + 1}{(x^2 + 1)^2} dx &= \int \left(\frac{3x+1}{x^2+1} + \frac{6x}{(x^2+1)^2} \right) dx \\ &= \int \frac{3x}{x^2+1} dx + \int \frac{1}{x^2+1} dx + \int \frac{6x}{(x^2+1)^2} dx \\ &= \int \frac{\frac{3}{2}}{u} du + \arctan x + \int \frac{3}{u^2} du \\ &= \frac{3}{2} \ln|u| + \arctan x - \frac{3}{u} + C \\ &= \frac{3}{2} \ln|x^2+1| + \arctan x - \frac{3}{x^2+1} + C \end{aligned}$$

$$w) \int \frac{x^4 + 5}{x^3 + 3x^2 - 10x} dx$$

$$\begin{array}{r} x-3 \\ \hline x^3 + 3x^2 - 10x \overline{) x^4 + 5} \\ \underline{x^4 + 3x^3 - 10x^2} \\ -3x^3 + 10x^2 + 5 \\ \underline{-3x^3 - 9x^2 + 30x} \\ 19x^2 - 30x + 5 \end{array}$$

$$\frac{x^4 + 5}{x^3 + 3x^2 - 10x} = x - 3 + \frac{19x^2 - 30x + 5}{x^3 + 3x^2 - 10x} = x - 3 + \frac{19x^2 - 30x + 5}{x(x+5)(x-2)}$$

$$\frac{19x^2 - 30x + 5}{x(x+5)(x-2)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2} = \frac{A(x+5)(x-2) + Bx(x-2) + Cx(x+5)}{x(x+5)(x-2)}$$

$$19x^2 - 30x + 5 = A(x+5)(x-2) + Bx(x-2) + Cx(x+5)$$

$$\text{If } x = 0, A = -\frac{1}{2}$$

$$\text{If } x = -5, B = 18$$

$$\text{If } x = 2, C = \frac{3}{2}$$

$$\begin{aligned} \int \frac{x^4 + 5}{x^3 + 3x^2 - 10x} dx &= \int \left(x - 3 - \frac{1}{2x} + \frac{18}{x+5} + \frac{3}{2} \frac{1}{x-2} \right) dx \\ &= \frac{1}{2} x^2 - 3x - \frac{1}{2} \ln|x| + 18 \ln|x+5| + \frac{3}{2} \ln|x-2| + C \end{aligned}$$

$$\begin{aligned}
 \text{x) } \int \csc^4 x \sqrt{\cot^5 x} dx &= \int (1 + \cot^2 x) \sqrt{\cot^5 x} \csc^2 x dx & u &= \cot x \\
 &= \int -(1 + u^2) u^{\frac{5}{2}} du & du &= -\csc^2 x dx \\
 &= -\int \left(u^{\frac{5}{2}} + u^{\frac{9}{2}}\right) du \\
 &= -\frac{2}{7} u^{\frac{7}{2}} - \frac{2}{11} u^{\frac{11}{2}} + C \\
 &= -\frac{2}{7} \cot^{\frac{7}{2}} x - \frac{2}{11} \cot^{\frac{11}{2}} x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{y) } \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^4 x dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2} \cos 2x\right)^2 dx \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x\right) dx \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{8} + \frac{1}{8} \cos 4x\right) dx \\
 &= \left[\frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \\
 &= \frac{\pi}{8} + \frac{1}{4} \frac{\sqrt{3}}{2} + \frac{1}{32} \left(\frac{-\sqrt{3}}{2}\right) - \left(\frac{-3\pi}{32}\right) - \frac{1}{4}(-1) - \frac{1}{32} 0 \\
 &= \frac{7\pi}{32} + \frac{7\sqrt{3}}{64} + \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{z) } \int_2^6 \frac{3x+4}{x^2-4x+20} dx &= \int_2^6 \frac{3x+4}{(x-2)^2+16} dx & x^2-4x+20 &= x^2-4x+4+20-4 \\
 &= \int_0^4 \frac{3(u+2)+4}{u^2+16} du & &= (x-2)^2+16 \\
 &= \int_0^4 \frac{3u}{u^2+16} du + \int_0^4 \frac{10}{u^2+16} du & u &= x-2 & x=2, u=0 \\
 &= \frac{3}{2} \int_{16}^{32} \frac{1}{v} dv + \left[\frac{10}{4} \arctan \frac{u}{4}\right]_0^4 & du &= dx & x=6, u=4 \\
 &= \left[\frac{3}{2} \ln |v|\right]_{16}^{32} + \frac{5}{2} \frac{\pi}{4} & v &= u^2+16 & u=0, v=16 \\
 &= \frac{3}{2} (\ln 32 - \ln 16) + \frac{5\pi}{8} & dv &= du & u=4, v=32 \\
 &= \frac{3}{2} (5 \ln 2 - 4 \ln 2) + \frac{5\pi}{8} \\
 &= \frac{3}{2} \ln 2 + \frac{5\pi}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{aa) } \int_1^e \frac{\ln x}{x^5} dx &= \left[\frac{-\ln x}{4x^4}\right]_1^e - \int_1^e \frac{-1}{4x^5} dx & u &= \ln x & v &= \frac{-1}{4x^4} dx \\
 &= \frac{-1}{4e^4} + 0 + \left[\frac{-1}{16x^4}\right]_1^e & du &= \frac{1}{x} dx & dv &= \frac{1}{x^5} dx \\
 &= \frac{-1}{4e^4} - \frac{1}{16e^4} + \frac{1}{16} \\
 &= \frac{1}{16} - \frac{5}{16e^4}
 \end{aligned}$$

5. It is estimated that t months from now the population of a certain town will be changing at a rate of $\frac{dP}{dt} = 2 + t\sqrt{t+1}$ people per month. The current population is 5000. What will be the population 9 months from now?

$$\begin{aligned}
 P(t) &= \int (2 + t\sqrt{t+1}) dt & v &= \sqrt{t+1} \\
 &= 2t + \int (v^2 - 1)v 2v dv & v^2 &= t+1 \\
 &= 2t + \int (2v^4 - 2v^2) dv & 2v dv &= dt \\
 &= 2t + \frac{2}{5}v^5 - \frac{2}{3}v^3 + C \\
 &= 2t + \frac{2}{5}(t+1)^{\frac{5}{2}} - \frac{2}{3}(t+1)^{\frac{3}{2}} + C \\
 P(0) &= \frac{2}{5} - \frac{2}{3} + C = 5000 \\
 C &= \frac{75004}{15} \\
 \therefore P(t) &= 2t + \frac{2}{5}(t+1)^{\frac{5}{2}} - \frac{2}{3}(t+1)^{\frac{3}{2}} + \frac{75004}{15} \\
 P(9) &= \frac{75274}{15} + \frac{100\sqrt{10}}{3} \approx 5124
 \end{aligned}$$

6. After t hours on the job, a factory worker can produce units at a rate of $\frac{dQ}{dt} = t \ln(10-t)$ units per hour. How many units does a worker who arrives at 8:00 A.M. produce during the first 3 hours?

$$\begin{aligned}
 \int_0^3 t \ln(10-t) dt &= \left[\frac{1}{2} t^2 \ln(10-t) \right]_0^3 + \frac{1}{2} \int_0^3 \frac{t^2}{10-t} dt & u &= \ln(10-t) & v &= \frac{1}{2} t^2 \\
 &= \frac{9}{2} \ln 7 + \frac{1}{2} \int_0^3 \left(-t - 10 + \frac{100}{10-t} \right) dt & du &= \frac{-1}{10-t} dt & dv &= t dt \\
 &= \frac{9}{2} \ln 7 + \left[-\frac{1}{4} t^2 - 5t - 50 \ln|10-t| \right]_0^3 & & & & \\
 &= \frac{9}{2} \ln 7 - \frac{9}{4} - 15 - 50 \ln 7 + 50 \ln 10 & & & & \\
 &= 50 \ln 10 - \frac{69}{4} - \frac{91}{2} \ln 7 & & & & \\
 &\approx 9.3 & & & & \\
 & & & & & \frac{-t-10}{t^2} \\
 & & & & & \frac{t^2-10t}{10t} \\
 & & & & & \frac{10t-100}{100}
 \end{aligned}$$

Hence a worker produces 9 units.

7. Find the average value of each function over the given interval.

a) $f(x) = x \ln x$ $[1, e]$

$$\begin{aligned} f_{avg} &= \frac{1}{e-1} \int_1^e x \ln x \, dx \\ &= \frac{1}{e-1} \left(\left[\frac{1}{2} x^2 \ln x \right]_1^e - \int_1^e \frac{1}{2} x \, dx \right) \\ &= \frac{1}{e-1} \left(\frac{1}{2} e^2 - \left[\frac{1}{4} x^2 \right]_1^e \right) \\ &= \frac{1}{e-1} \left(\frac{1}{2} e^2 - \frac{1}{4} e^2 + \frac{1}{4} \right) \\ &= \frac{e^2 + 1}{4(e-1)} \end{aligned}$$

$$\begin{aligned} u &= \ln x & v &= \frac{1}{2} x^2 \\ du &= \frac{1}{x} dx & dv &= x dx \end{aligned}$$

b) $f(x) = \frac{x}{\sqrt{x+1}}$ $[8, 24]$

$$\begin{aligned} f_{avg} &= \frac{1}{24-8} \int_8^{24} \frac{x}{\sqrt{x+1}} \, dx \\ &= \frac{1}{16} \int_3^5 \frac{(v^2-1)2v}{v} \, dv \\ &= \frac{1}{16} \int_3^5 (2v^2-2) \, dv \\ &= \frac{1}{16} \left[\frac{2}{3} v^3 - 2v \right]_3^5 \\ &= \frac{1}{16} \left(\frac{250}{3} - 10 - 18 + 6 \right) \\ &= \frac{23}{6} \end{aligned}$$

$$\begin{aligned} v &= \sqrt{x+1} \\ v^2 &= x+1 \\ 2v \, dv &= dx \\ x=8, v &= 3 \\ x=24, v &= 5 \end{aligned}$$

c) $f(x) = \frac{5}{x^2 - 4x + 3}$ $[4, 6]$

$$\begin{aligned} f_{avg} &= \frac{1}{6-4} \int_4^6 \frac{5}{x^2 - 4x + 3} \, dx \\ &= \frac{1}{2} \int_4^6 \left(\frac{\frac{5}{2}}{x-3} - \frac{\frac{5}{2}}{x-1} \right) \, dx \\ &= \frac{1}{2} \left[\frac{5}{2} \ln|x-3| - \frac{5}{2} \ln|x-1| \right]_4^6 \\ &= \frac{5}{4} \ln 3 - \frac{5}{4} \ln 5 + \frac{5}{4} \ln 3 \\ &= \frac{5}{2} \ln 3 - \frac{5}{4} \ln 5 \end{aligned}$$

$$\begin{aligned} \frac{5}{x^2 - 4x + 3} &= \frac{5}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1} \\ &= \frac{A(x-1) + B(x-3)}{(x-3)(x-1)} \\ \therefore 5 &= (A+B)x - A - 3B \\ x: 0 &= A+B \\ : 5 &= -A - 3B \\ B &= \frac{-5}{2}, \quad A = \frac{5}{2} \end{aligned}$$

8. Suppose the number of items a new worker on an assembly line produces daily after t days on the job is given by $I(t) = 35 \ln(t+1)$. Find the average number of items produced daily by this employee after 10 days.

$$\begin{aligned}
 I_{\text{avg}} &= \frac{1}{10-0} \int_0^{10} 35 \ln(t+1) dt & u &= \ln x & v &= \frac{1}{2} x^2 \\
 &= \frac{35}{10} \left([t \ln|t+1|]_0^{10} - \int_0^{10} \frac{t}{t+1} dt \right) & du &= \frac{1}{x} dx & dv &= x dx \\
 &= \frac{7}{2} \left(10 \ln 11 - \int_0^{10} \left(1 - \frac{1}{t+1} \right) dt \right) \\
 &= \frac{7}{2} \left(10 \ln 11 - [t - \ln|t+1|]_0^{10} \right) \\
 &= \frac{7}{2} (10 \ln 11 - 10 + \ln 11) \\
 &= \frac{77}{2} \ln 11 - 35 \\
 &\approx 57.3 \text{ items}
 \end{aligned}$$

9. Determine whether the following improper integral converges or diverges.

$$\begin{aligned}
 \text{a) } \int_7^{\infty} \frac{1}{(x+2)^{\frac{5}{3}}} dx &= \lim_{t \rightarrow \infty} \int_7^t (x+2)^{-\frac{5}{3}} dx \\
 &= \lim_{t \rightarrow \infty} \left[\frac{-2}{3} (x+2)^{-\frac{2}{3}} \right]_7^t \\
 &= \lim_{t \rightarrow \infty} \left(\frac{-2}{3(x+2)^{\frac{2}{3}}} + \frac{2}{81} \right) \\
 &= \frac{2}{81}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } \int_{-\infty}^3 \frac{x+1}{x^2+1} dx &= \lim_{t \rightarrow -\infty} \int_t^3 \frac{x+1}{x^2+1} dx & u &= x^2+1 \\
 &= \lim_{t \rightarrow -\infty} \left(\int_t^3 \frac{x}{x^2+1} dx + \int_t^3 \frac{1}{x^2+1} dx \right) & du &= 2x dx \\
 &= \lim_{t \rightarrow -\infty} \left(\int_{x=t}^{x=3} \frac{\frac{1}{2}}{u} du + [\arctan x]_t^3 \right) \\
 &= \lim_{t \rightarrow -\infty} \left(\left[\frac{1}{2} \ln|u| \right]_{x=t}^{x=3} + \arctan 3 - \arctan t \right) \\
 &= \lim_{t \rightarrow -\infty} \left(\left[\frac{1}{2} \ln|x^2+1| \right]_t^3 + \arctan 3 - \arctan t \right) \\
 &= \lim_{t \rightarrow -\infty} \left(\frac{1}{2} \ln 10 - \frac{1}{2} \ln|t^2+1| + \arctan 3 - \arctan t \right) \\
 &= -\infty
 \end{aligned}$$

Thus $\int_{-\infty}^3 \frac{x+1}{x^2+1} dx$ diverges.

$$c) \int_{-2}^{\infty} \frac{x}{(x-1)^2} dx = \int_{-2}^1 \frac{x}{(x-1)^2} dx + \int_1^2 \frac{x}{(x-1)^2} dx + \int_2^{\infty} \frac{x}{(x-1)^2} dx$$

$$\int_2^{\infty} \frac{x}{(x-1)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{x}{(x-1)^2} dx$$

$$= \lim_{t \rightarrow \infty} \int_2^t \left(\frac{1}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$= \lim_{t \rightarrow \infty} \left[\ln|x-1| - \frac{1}{x-1} \right]_2^t$$

$$= \lim_{t \rightarrow \infty} \left(\ln|t-1| - \frac{1}{t-1} - \ln 1 + 1 \right)$$

$$= \infty$$

Thus $\int_{-2}^{\infty} \frac{x}{(x-1)^2} dx$ diverges

$$d) \int_1^6 \frac{3x}{\sqrt[3]{x^2-9}} dx = \int_1^3 \frac{3x}{\sqrt[3]{x^2-9}} dx + \int_3^6 \frac{3x}{\sqrt[3]{x^2-9}} dx$$

$$\int_1^3 \frac{3x}{\sqrt[3]{x^2-9}} dx = \lim_{t \rightarrow 3^-} \int_1^t \frac{3x}{\sqrt[3]{x^2-9}} dx$$

$$= \lim_{t \rightarrow 3^-} \int_{x=1}^{x=t} \frac{\frac{3}{2}}{\sqrt[3]{u}} du$$

$$= \lim_{t \rightarrow 3^-} \left[\frac{9}{4} u^{\frac{2}{3}} \right]_{x=1}^{x=t}$$

$$= \lim_{t \rightarrow 3^-} \left[\frac{9}{4} (x^2-9)^{\frac{2}{3}} \right]_1^t$$

$$= \lim_{t \rightarrow 3^-} \left(\frac{9}{4} (t^2-9)^{\frac{2}{3}} - 9 \right)$$

$$= -9$$

$$\int_3^6 \frac{3x}{\sqrt[3]{x^2-9}} dx = \lim_{t \rightarrow 3^+} \int_t^6 \frac{3x}{\sqrt[3]{x^2-9}} dx$$

$$= \lim_{t \rightarrow 3^+} \left[\frac{9}{4} (x^2-9)^{\frac{2}{3}} \right]_t^6$$

$$= \lim_{t \rightarrow 3^+} \left(\frac{81}{4} - \frac{9}{4} (t^2-9)^{\frac{2}{3}} \right)$$

$$= \frac{81}{4}$$

$$\text{Thus } \int_1^6 \frac{3x}{\sqrt[3]{x^2-9}} dx = -9 + \frac{81}{4} = \frac{45}{4}$$

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$= \frac{A(x-1) + B}{(x-1)^2}$$

$$x = Ax - A + B$$

$$A = 1$$

$$B = 1$$

10. Estimate $\int_1^9 \ln(1+x) dx$ to 6 decimal places with $n = 4$ using

a) right endpoint approximation

$$\begin{aligned} R_4 &= \sum_{i=1}^4 f(x_i) \Delta x = (f(x_1) + f(x_2) + f(x_3) + f(x_4)) \Delta x & \Delta x &= \frac{b-a}{n} = 2 \\ &= 2(\ln(4) + \ln(6) + \ln(8) + \ln(10)) & x_i &= a + i\Delta x = 1 + 2i \\ &\approx 15.120161 \end{aligned}$$

b) left endpoint approximation

$$\begin{aligned} L_4 &= \sum_{i=1}^4 f(x_{i-1}) \Delta x = (f(x_0) + f(x_1) + f(x_2) + f(x_3)) \Delta x \\ &= 2(\ln(2) + \ln(4) + \ln(6) + \ln(8)) \\ &\approx 11.901285 \end{aligned}$$

c) midpoint rule

$$\begin{aligned} M_4 &= \sum_{i=1}^4 f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x & \frac{x_{i-1} + x_i}{2} &= \frac{1 + 2(i-1) + 1 + 2i}{2} \\ &= (f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)) \Delta x & &= 2i \\ &= 2(\ln(3) + \ln(5) + \ln(7) + \ln(9)) \\ &\approx 13.702370 \end{aligned}$$

d) trapezoidal rule

$$\begin{aligned} T_4 &= \sum_{i=1}^4 (f(x_{i-1}) + f(x_i)) \frac{\Delta x}{2} \\ &= (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) \frac{\Delta x}{2} \\ &= (\ln(2) + 2\ln(4) + 2\ln(6) + 2\ln(8) + \ln(10)) \\ &\approx 13.510723 \end{aligned}$$

e) Simpson's rule

$$\begin{aligned} S_4 &= \sum_{i=1}^2 (f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})) \frac{\Delta x}{3} \\ &= (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) \frac{\Delta x}{3} \\ &= (\ln(2) + 4\ln(4) + 2\ln(6) + 4\ln(8) + \ln(10)) \\ &\approx 13.628130 \end{aligned}$$

- f) Find the maximum error of estimate for the midpoint rule.

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{x+1}$$

$$f''(x) = \frac{-1}{(x+1)^2}$$

$$\text{Thus } |f''(x)| \leq \frac{1}{4} \text{ on } [1, 9]$$

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{\frac{1}{4}8^3}{24 \cdot 4^2} = \frac{1}{3}$$

- g) Find the maximum error of estimate for the trapezoidal rule.

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{\frac{1}{4}8^3}{12 \cdot 4^2} = \frac{2}{3}$$

- h) Find the maximum error of estimate for the Simpson's rule.

$$f'''(x) = \frac{2}{(x+1)^3}$$

$$f^{(iv)}(x) = \frac{-6}{(x+1)^4}$$

$$\text{Thus } |f^{(iv)}(x)| \leq \frac{3}{8} \text{ on } [1, 9]$$

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{\frac{3}{8}8^5}{180 \cdot 4^4} = \frac{4}{15}$$

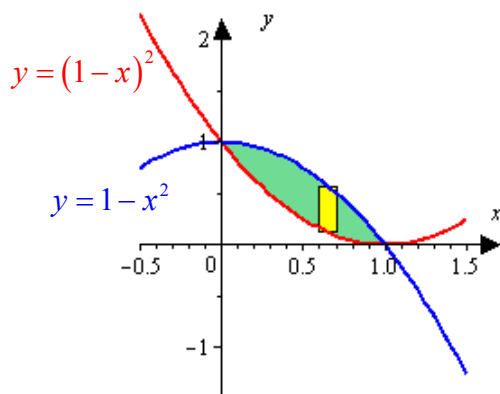
- i) Find the exact value of
- $\int_1^9 \ln(1+x) dx$
- .

$$\begin{aligned} \int_1^9 \ln(1+x) dx &= [x \ln(1+x)]_1^9 - \int_1^9 \frac{x}{1+x} dx \\ &= 9 \ln 10 - \ln 2 - \int_1^9 \left(1 - \frac{1}{1+x}\right) dx \\ &= 9 \ln 10 - \ln 2 - [x - \ln|x+1|]_1^9 \\ &= 9 \ln 10 - \ln 2 - 9 + \ln 10 + 1 - \ln 2 \\ &= 10 \ln 10 - 2 \ln 2 - 8 \\ &\approx 13.639557 \end{aligned}$$

$$\begin{aligned} u &= \ln(1+x) & v &= x \\ du &= \frac{1}{1+x} dx & dv &= dx \end{aligned}$$

11. Find the area of the region R if

a) R is bounded by the curves $y = (x-1)^2$ and $y = 1-x^2$.



Points of intersection: $(1-x)^2 = 1-x^2$
 $1-2x+x^2 = 1-x^2$
 $2x^2-2x=0$
 $2x(x-1)=0$
 $x=0,1$

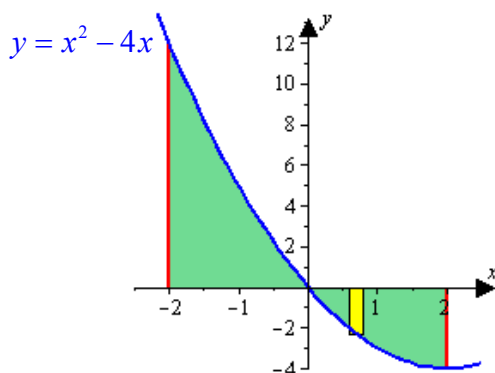
$$A = \int_0^1 (1-x^2 - (x-1)^2) dx$$

$$= \left[x - \frac{1}{3}x^3 - \frac{1}{3}(x-1)^3 \right]_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{3}$$

$$= \frac{1}{3}$$

b) R is bounded by the curves $y = x^2 - 4x$, the x-axis, $x = -2$ and $x = 2$.



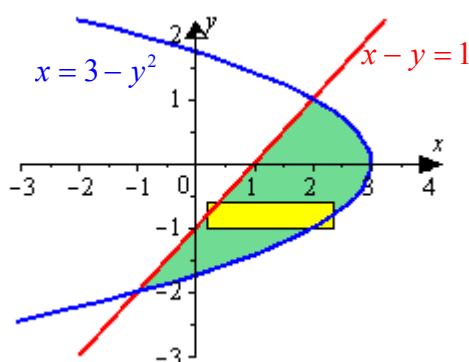
$$A = \int_{-2}^0 (x^2 - 4x) dx + \int_0^2 (-x^2 + 4x) dx$$

$$= \left[\frac{1}{3}x^3 - 2x^2 \right]_{-2}^0 + \left[-\frac{1}{3}x^3 + 2x^2 \right]_0^2$$

$$= -\frac{8}{3} - 8 - \frac{8}{3} + 8$$

$$= 16$$

c) R is bounded by the curves $x = 3 - y^2$ and $x - y = 1$.



Points of intersection: $3 - y^2 = y + 1$
 $y^2 + y - 2 = 0$
 $(y+2)(y-1) = 0$
 $y = -2, 1$

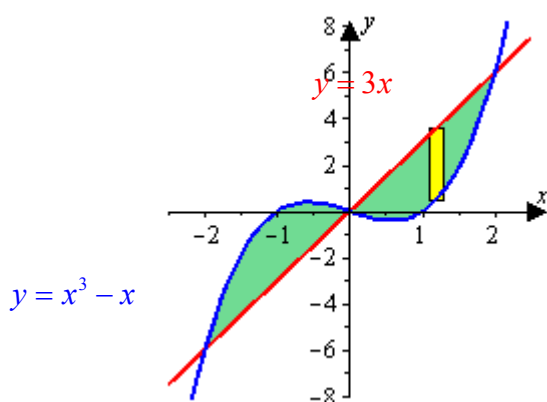
$$A = \int_{-2}^1 (3 - y^2 - (y+1)) dy$$

$$= \left[2y - \frac{1}{3}y^3 - \frac{1}{2}y^2 \right]_{-2}^1$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2} \right) - \left(-4 + \frac{8}{3} - 2 \right)$$

$$= \frac{9}{2}$$

d) R is bounded by the curve $y = x^3 - x$ and $y = 3x$.



Points of intersection: $x^3 - x = 3x$

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$$x = -2, 0, 2$$

$$A = \int_{-2}^0 (x^3 - x - 3x) dx + \int_0^2 (3x - x^3 + x) dx$$

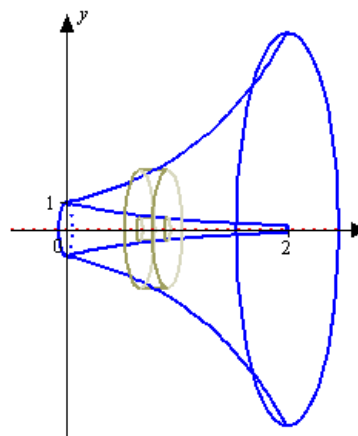
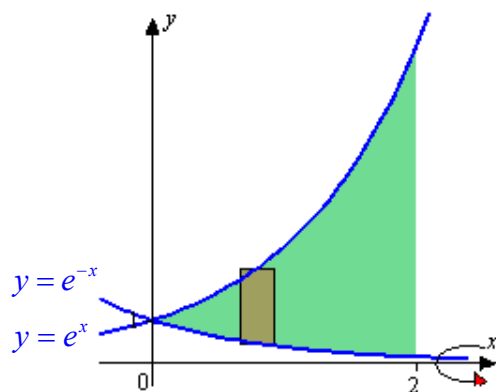
$$= \left[\frac{1}{4}x^4 - 2x^2 \right]_{-2}^0 + \left[-\frac{1}{4}x^4 + 2x^2 \right]_0^2$$

$$= 0 - (4 - 8) + (-4 + 8)$$

$$= 8$$

12. Consider the region R bounded by $y = e^x$, $y = e^{-x}$ and $x = 2$.

a) Find the volume of the solid obtained by rotating the region about the x -axis. Sketch the region and a typical disk, washer of shell.



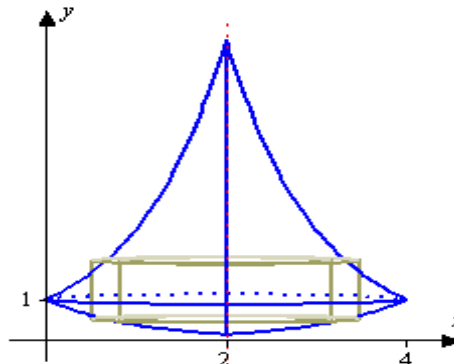
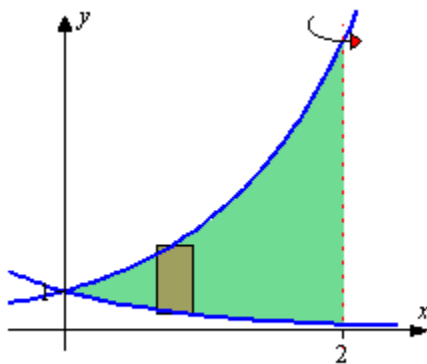
$$V_w = \int_0^2 (\pi(e^x)^2 - \pi(e^{-x})^2) dx$$

$$= \int_0^2 (\pi e^{2x} - \pi e^{-2x}) dx$$

$$= \left[\frac{\pi}{2} e^{2x} + \frac{\pi}{2} e^{-2x} \right]_0^2$$

$$= \pi \left(\frac{e^4}{2} + \frac{1}{2e^4} - 1 \right)$$

- b) Find the volume of the solid obtained by rotating the region about the line $x=2$. Sketch the region and a typical disk, washer of shell.



$$V_S = \int_0^2 2\pi(2-x)(e^x - e^{-x}) dx$$

$$= 2\pi \int_0^2 (2e^x - 2e^{-x} - x(e^x - e^{-x})) dx$$

$$= 2\pi \left([2e^x + 2e^{-x}]_0^2 - [x(e^x + e^{-x})]_0^2 + \int_0^2 (e^x + e^{-x}) dx \right)$$

$$= 2\pi \left(2e^2 + 2e^{-2} - 4 - 2e^2 - 2e^{-2} + [e^x - e^{-x}]_0^2 \right)$$

$$= 2\pi(e^2 - e^{-2} - 4)$$

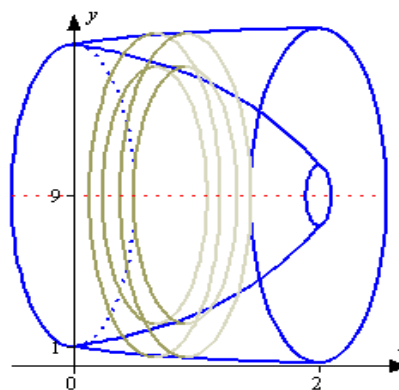
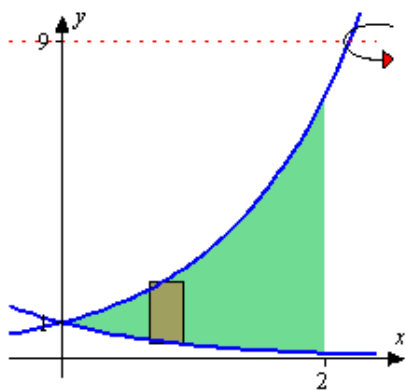
$$u = x$$

$$du = dx$$

$$v = e^x + e^{-x}$$

$$dv = (e^x - e^{-x}) dx$$

- c) Find the volume of the solid obtained by rotating the region about the line $y=9$. Sketch the region and a typical disk, washer of shell.



$$V_W = \int_0^2 \left(\pi(9 - e^{-x})^2 - \pi(9 - e^x)^2 \right) dx$$

$$= \pi \int_0^2 (81 - 18e^{-x} + e^{-2x} - 81 + 18e^x - e^{2x}) dx$$

$$= \pi \int_0^2 (-18e^{-x} + e^{-2x} + 18e^x - e^{2x}) dx$$

$$= \pi \left(18e^{-x} - \frac{1}{2}e^{-2x} + 18e^x - \frac{1}{2}e^{2x} \right) \Big|_0^2$$

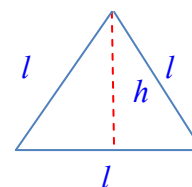
$$= \pi \left(\frac{18}{e^2} - \frac{1}{2e^4} + 18e^2 - \frac{1}{2}e^4 - 35 \right)$$

- d) Find the volume of the solid whose cross-sections perpendicular to the x -axis are squares.

$$\begin{aligned}
 V &= \int_0^2 A(x) dx & A(x) &= l^2 \\
 &= \int_0^2 (e^x - e^{-x})^2 dx & &= (e^x - e^{-x})^2 \\
 &= \int_0^2 (e^{2x} - 2 + e^{-2x}) dx \\
 &= \left[\frac{1}{2} e^{2x} - 2x - \frac{1}{2} e^{-2x} \right]_0^2 \\
 &= \frac{1}{2} e^4 - 4 - \frac{1}{2e^4}
 \end{aligned}$$

- e) Find the volume of the solid whose cross-sections perpendicular to the x -axis are equilateral triangles.

$$\begin{aligned}
 V &= \int_0^2 A(x) dx & A(x) &= \frac{1}{2} bh \\
 &= \int_0^2 \frac{\sqrt{3}}{4} (e^x - e^{-x})^2 dx & &= \frac{1}{2} l \frac{\sqrt{3}}{2} l \\
 &= \frac{\sqrt{3}}{4} \int_0^2 (e^{2x} - 2 + e^{-2x}) dx & &= \frac{\sqrt{3}}{4} l^2 \\
 &= \frac{\sqrt{3}}{4} \left[\frac{1}{2} e^{2x} - 2x - \frac{1}{2} e^{-2x} \right]_0^2 & &= \frac{\sqrt{3}}{4} (e^x - e^{-x})^2 \\
 &= \frac{\sqrt{3}}{8} e^4 - \sqrt{3} - \frac{\sqrt{3}}{8e^4}
 \end{aligned}$$

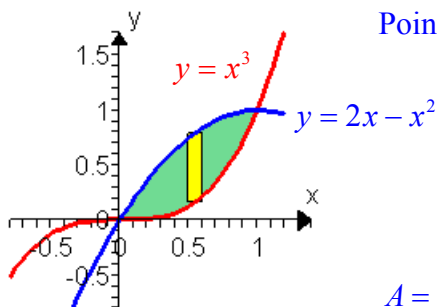


$$\tan 60^\circ = \frac{h}{\frac{l}{2}}$$

$$h = \frac{l}{2} \sqrt{3} = \frac{\sqrt{3}}{2} l$$

13. Consider the region R in the first quadrant bounded by the curves $y = x^3$, $y = 2x - x^2$.

- a) Sketch the region R and find the area.

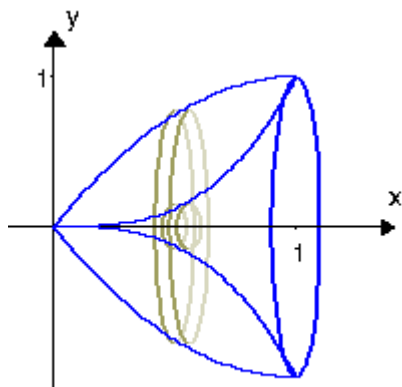


Points of intersection:

$$\begin{aligned}
 x^3 &= 2x - x^2 \\
 x^3 + x^2 - 2x &= 0 \\
 x(x^2 + x - 2) &= 0 \\
 x(x+2)(x-1) &= 0 \\
 x &= -2, 0, 1
 \end{aligned}$$

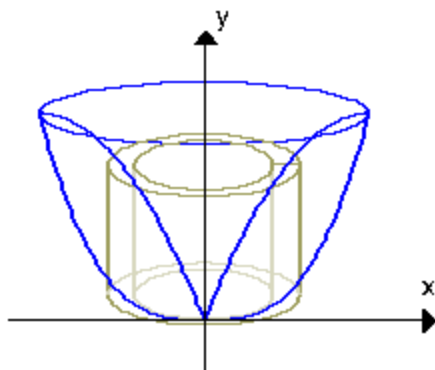
$$\begin{aligned}
 A &= \int_0^1 (2x - x^2 - x^3) dx \\
 &= \left[x^2 - \frac{1}{3} x^3 - \frac{1}{4} x^4 \right]_0^1 \\
 &= 1 - \frac{1}{3} - \frac{1}{4} \\
 &= \frac{5}{12}
 \end{aligned}$$

- b) Find the volume obtained by rotating R about the x -axis. Sketch the region and a typical disk, washer of shell.



$$\begin{aligned} V_w &= \int_0^1 \left(\pi(2x-x^2)^2 - \pi(x^3)^2 \right) dx \\ &= \pi \int_0^1 (x^4 - 4x^3 + 4x^2 - x^6) dx \\ &= \pi \left[\frac{1}{5}x^5 - x^4 + \frac{4}{3}x^3 - \frac{1}{7}x^7 \right]_0^1 \\ &= \pi \left(\frac{1}{5} - 1 + \frac{4}{3} - \frac{1}{7} \right) \\ &= \frac{41\pi}{105} \end{aligned}$$

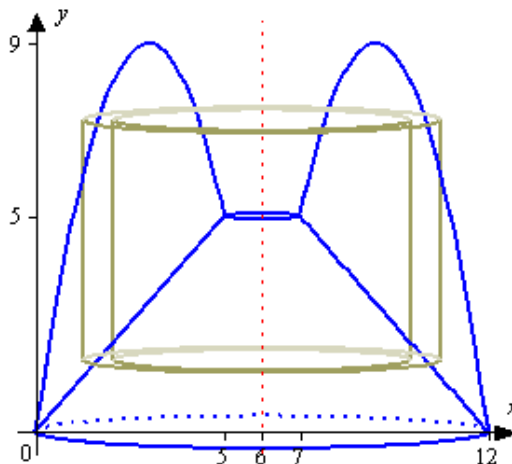
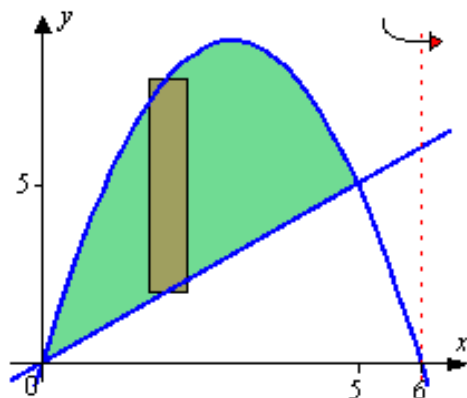
- c) Find the volume obtained by rotating R about the y -axis. Sketch the region and a typical disk, washer of shell.



$$\begin{aligned} V_s &= \int_0^1 2\pi x(2x-x^2-x^3) dx \\ &= 2\pi \int_0^1 (2x^2 - x^3 - x^4) dx \\ &= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 \right]_0^1 \\ &= 2\pi \left(\frac{2}{3} - \frac{1}{4} - \frac{1}{5} \right) \\ &= \frac{13\pi}{30} \end{aligned}$$

14. Consider the region R bounded by $y = 6x - x^2$ and $y = x$. Find the volume of the solid if cross-sections perpendicular to the x -axis are

- a) Find the volume of the solid obtained by rotating the region about the line $x = 6$. Sketch the region and a typical disk, washer of shell.



Points of intersection:

$$6x - x^2 = x$$

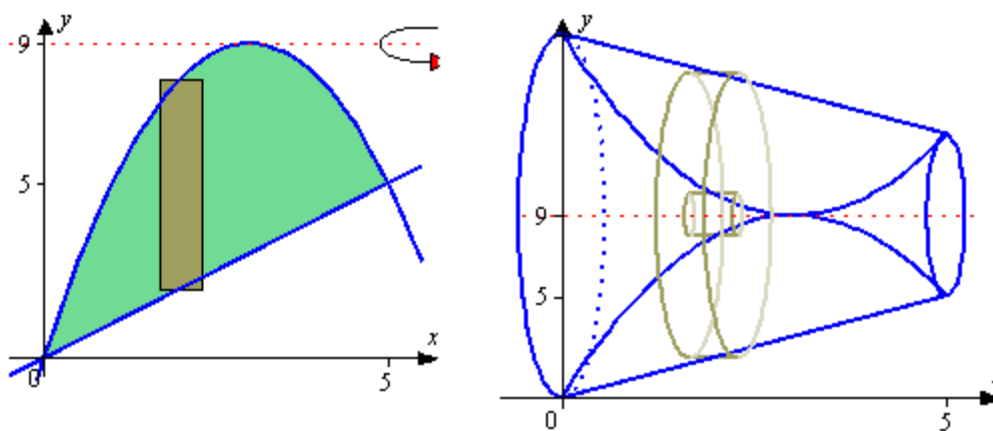
$$x^2 - 5x = 0$$

$$x(x-5) = 0$$

$$x = 0, 5$$

$$\begin{aligned} V_s &= 2\pi \int_0^5 (6-x)(6x-x^2-x) dx \\ &= 2\pi \int_0^5 (6-x)(5x-x^2) dx \\ &= 2\pi \int_0^5 (x^3 - 11x^2 + 30x) dx \\ &= 2\pi \left[\frac{1}{4}x^4 - \frac{11}{3}x^3 + 15x^2 \right]_0^5 \\ &= 2\pi \left(\frac{625}{4} - \frac{1375}{3} + 375 \right) \\ &= \frac{875\pi}{6} \end{aligned}$$

- b) Find the volume of the solid obtained by rotating the region about the line $y = 9$. Sketch the region and a typical disk, washer of shell.



$$\begin{aligned} V_w &= \pi \int_0^5 \left((9-x)^2 - (9-(6x-x^2))^2 \right) dx \\ &= \pi \int_0^5 (x^2 - 18x + 81 - x^4 + 12x^3 - 54x + 108x - 81) dx \\ &= \pi \int_0^5 (-x^4 + 12x^3 - 53x^2 + 90x) dx \\ &= \pi \left[-\frac{1}{5}x^5 + 3x^4 - \frac{53}{3}x^3 + 45x^2 \right]_0^5 \\ &= \pi \left(-625 + 1875 - \frac{6625}{3} + 1125 \right) \\ &= \frac{500\pi}{3} \end{aligned}$$

- c) Find the volume of the solid whose base is R and cross-sections perpendicular to the x -axis are squares.

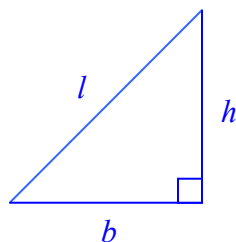
$$\begin{aligned} V &= \int_0^5 (5x - x^2)^2 dx & A(x) &= l^2 \\ &= \int_0^5 (x^4 - 10x^3 + 25x^2) dx & &= (6x - x^2 - x)^2 \\ &= \left[\frac{1}{5}x^5 - \frac{5}{2}x^4 + \frac{25}{3}x^3 \right]_0^5 & &= (5x - x^2)^2 \\ &= 625 - \frac{3125}{2} + \frac{3125}{3} \\ &= \frac{625}{6} \end{aligned}$$

- d) Find the volume of the solid whose base is R and cross-sections perpendicular to the x -axis are semi-circles.

$$\begin{aligned} V &= \int_0^5 \frac{\pi}{8} (5x - x^2)^2 dx \\ &= \frac{\pi}{8} \int_0^5 (x^4 - 10x^3 + 25x^2) dx \\ &= \frac{\pi}{8} \left[\frac{1}{5}x^5 - \frac{5}{2}x^4 + \frac{25}{3}x^3 \right]_0^5 \\ &= \frac{\pi}{8} \left(625 - \frac{3125}{2} + \frac{3125}{3} \right) \\ &= \frac{625\pi}{48} \end{aligned}$$

$$\begin{aligned} A(x) &= \frac{1}{2} \pi r^2 \\ &= \frac{1}{2} \pi \left(\frac{6x - x^2 - x}{2} \right)^2 \\ &= \frac{\pi}{8} (5x - x^2)^2 \end{aligned}$$

- e) Find the volume of the solid whose base is R and cross-sections perpendicular to the x -axis are isosceles right triangles with the hypotenuse as the base.

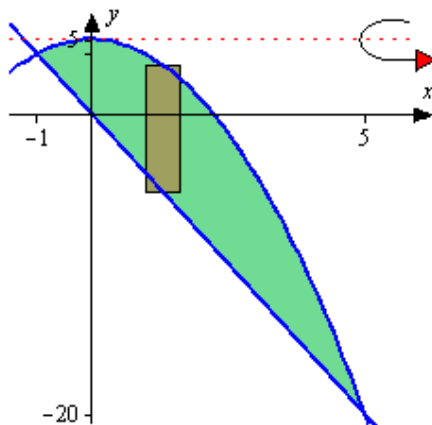


$$\begin{aligned} l^2 &= \sqrt{b^2 + h^2} = \sqrt{2b^2} \\ l &= \sqrt{2}b \\ b &= \frac{\sqrt{2}}{2}l \end{aligned}$$

$$\begin{aligned} A(x) &= \frac{1}{2}bh = \frac{1}{2} \frac{\sqrt{2}}{2} l \frac{\sqrt{2}}{2} l = \frac{1}{4}l^2 \\ &= \frac{1}{4}(6x - x^2 - x)^2 \\ &= \frac{1}{4}(5x - x^2)^2 \\ V &= \int_0^5 \frac{1}{4}(5x - x^2)^2 dx \\ &= \frac{1}{4} \int_0^5 (x^4 - 10x^3 + 25x^2) dx \\ &= \frac{1}{4} \left[\frac{1}{5}x^5 - \frac{5}{2}x^4 + \frac{25}{3}x^3 \right]_0^5 \\ &= \frac{1}{4} \left(625 - \frac{3125}{2} + \frac{3125}{3} \right) \\ &= \frac{625}{24} \end{aligned}$$

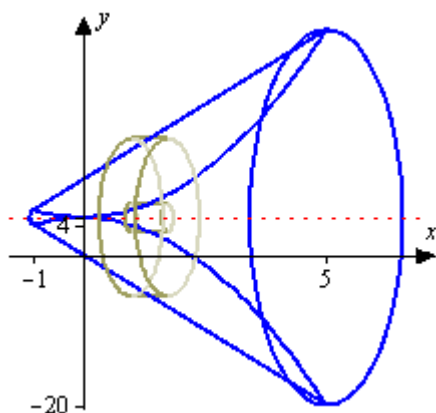
15. Consider the region R bounded by $y = 5 - x^2$ and $y = -4x$.

- a) Find the volume obtained by rotating R about the line $y = 5$. Sketch the region and a typical disk, washer of shell.



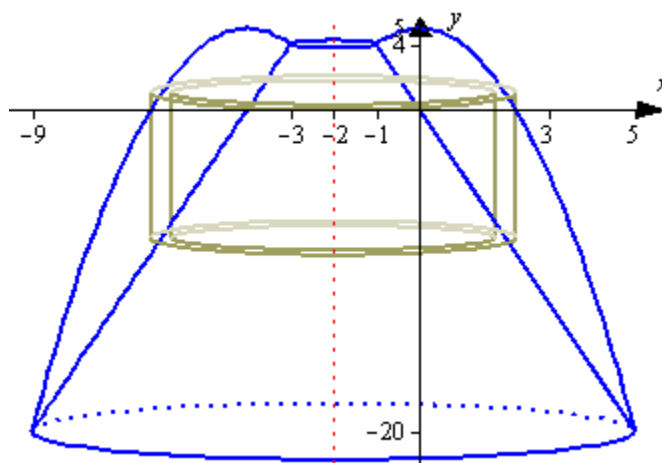
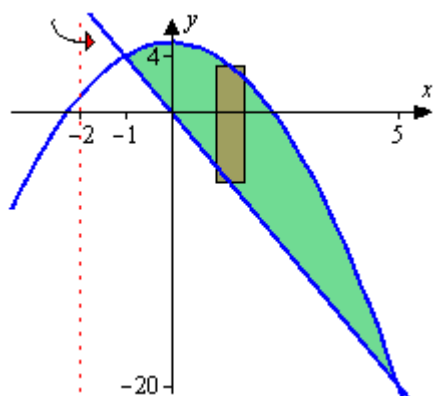
Points of intersection:

$$\begin{aligned} 5 - x^2 &= -4x \\ x^2 - 4x - 5 &= 0 \\ (x - 5)(x + 1) &= 0 \\ x &= -1, 5 \end{aligned}$$



$$\begin{aligned}
 V_w &= \int_{-1}^5 \left(\pi(5 - (-4x))^2 - \pi(5 - (5 - x^2))^2 \right) dx \\
 &= \int_{-1}^5 \pi(25 + 40x + 16x^2 - x^4) dx \\
 &= \pi \left[25x + 20x^2 + \frac{16}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^5 \\
 &= \pi \left(125 + 500 + \frac{2000}{3} - 625 + 25 - 20 + \frac{16}{3} - \frac{1}{5} \right) \\
 &= \frac{3384\pi}{5}
 \end{aligned}$$

- b) Find the volume obtained by rotating R about the line $x = -2$. Sketch the region and a typical disk, washer of shell.



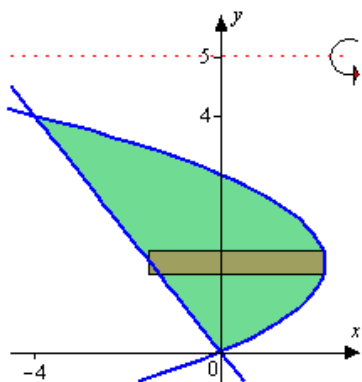
$$\begin{aligned}
 V_s &= \int_{-1}^5 2\pi(x+2)(5-x^2+4x) dx \\
 &= \int_{-1}^5 2\pi(-x^3 + 2x^2 + 13x + 10) dx \\
 &= 2\pi \left[-\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{13}{2}x^2 + 10x \right]_{-1}^5 \\
 &= 2\pi \left(-\frac{625}{4} + \frac{250}{3} + \frac{325}{2} + 50 + \frac{1}{4} + \frac{2}{3} - \frac{13}{2} + 10 \right) \\
 &= 288\pi
 \end{aligned}$$

- c) Find the volume of the solid whose base is R and if cross-sections perpendicular to the x -axis are quarter circles, whose radius is on the base.

$$\begin{aligned}
 A(x) &= \frac{1}{4}\pi r^2 & V &= \int_{-1}^5 \frac{\pi}{4}(5-x^2+4x)^2 dx \\
 &= \frac{\pi}{4}(5-x^2+4x)^2 & &= \int_{-1}^5 \frac{\pi}{4}(x^4 - 8x^3 + 6x^2 + 40x + 25) dx \\
 & & &= \frac{\pi}{4} \left[\frac{1}{5}x^5 - 2x^4 + 2x^3 + 20x^2 + 25x \right]_{-1}^5 \\
 & & &= \frac{\pi}{4} \left(625 - 1250 + 250 + 500 + 125 + \frac{1}{5} + 2 + 2 - 20 + 25 \right) \\
 & & &= \frac{324\pi}{5}
 \end{aligned}$$

16. Consider the region R bounded by $x = 3y - y^2$ and $y = -x$.

- a) Find the volume obtained by rotating R about the line $y = 5$. Sketch the region and a typical disk, washer of shell.



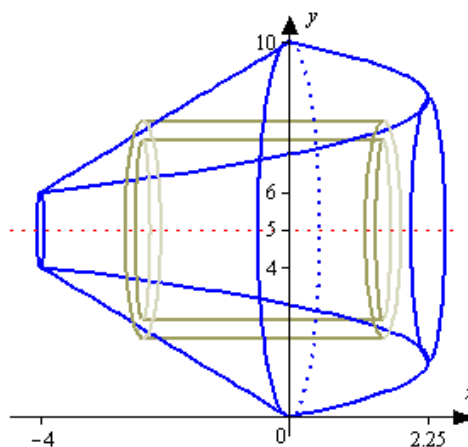
Pts of Intersection:

$$3y - y^2 = -y$$

$$4y - y^2 = 0$$

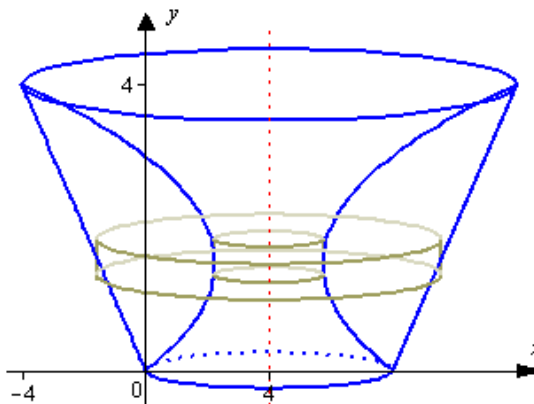
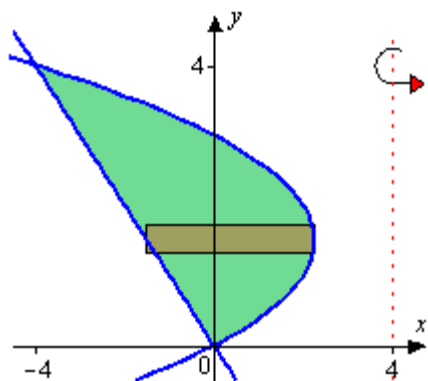
$$y(4 - y) = 0$$

$$y = 0, 4$$



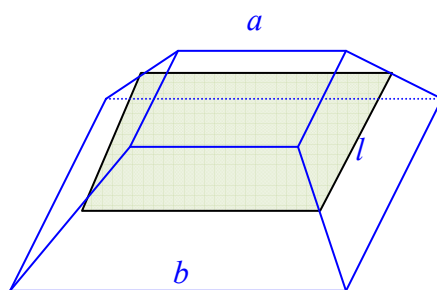
$$\begin{aligned} V_s &= 2\pi \int_0^4 (5 - y)(3y - y^2 - (-y)) dy \\ &= 2\pi \int_0^4 (y^3 - 9y^2 + 20y) dy \\ &= 2\pi \left[\frac{1}{4}y^4 - 3y^3 + 10y^2 \right]_0^4 \\ &= 2\pi(64 - 192 + 160) \\ &= 64\pi \end{aligned}$$

- b) Find the volume obtained by rotating R about the line $x = 4$. Sketch the region and a typical disk, washer of shell.

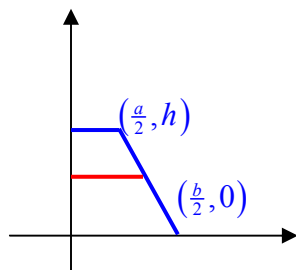


$$\begin{aligned}
 V_w &= \pi \int_0^4 \left((4 - (-y))^2 - (4 - (3y - y^2))^2 \right) dy \\
 &= \pi \int_0^4 (y^2 - 8y + 16 - y^4 + 6y^3 - 17y^2 + 24y - 16) dy \\
 &= \pi \int_0^4 (-y^4 + 6y^3 - 16y^2 + 32y) dy \\
 &= \pi \left[-\frac{1}{5}y^5 + \frac{3}{2}y^4 - \frac{16}{3}y^3 + 16y^2 \right]_0^4 \\
 &= \pi \left(-\frac{1024}{5} + 384 - \frac{1024}{3} + 256 \right) \\
 &= \frac{1408}{15} \pi
 \end{aligned}$$

17. Find the volume of a frustum of a pyramid with square base of side b , square top of side a , and height h .



Using rectangular cross sections,



$$\begin{aligned}
 m &= -\frac{2h}{b-a} \\
 y &= -\frac{2h}{b-a}x + \frac{bh}{b-a} \\
 x &= \frac{a-b}{2h}y + \frac{b}{2}
 \end{aligned}$$

$$\begin{aligned}
 A(y) &= l^2 = (2x)^2 = 4\left(\frac{a-b}{2h}y + \frac{b}{2}\right)^2 \\
 V &= \int_0^h 4\left(\frac{a-b}{2h}y + \frac{b}{2}\right)^2 dy \\
 &= \left[\frac{4}{3} \left(\frac{a-b}{2h}y + \frac{b}{2}\right)^3 \left(\frac{2h}{a-b}\right) \right]_0^h \\
 &= \frac{4}{3} \left(\frac{a}{2}\right)^3 \left(\frac{2h}{a-b}\right) - \frac{4}{3} \left(\frac{b}{2}\right)^3 \left(\frac{2h}{a-b}\right) \\
 &= \frac{8h}{3(a-b)^3} (a^3 - b^3) \\
 &= \frac{h(a-b)(a^2 + ab + b^2)}{3(a-b)} \\
 &= \frac{1}{3}h(a^2 + ab + b^2)
 \end{aligned}$$

18. Find the length of the curve given by $y = \frac{2}{3}(x^2 - 1)^{\frac{3}{2}}$ between $x = 1$ and $x = 3$.

$$\begin{aligned}
 L &= \int ds = \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx & \frac{dx}{dy} &= \frac{2}{3} \frac{3}{2} (x^2 - 1)^{\frac{1}{2}} 2x = 2x\sqrt{x^2 - 1} \\
 &= \int_1^3 \sqrt{1 + (2x\sqrt{x^2 - 1})^2} dx \\
 &= \int_1^3 \sqrt{1 + 4x^4 - 4x^2} dx \\
 &= \int_1^3 \sqrt{(2x^2 - 1)^2} dx \\
 &= \int_1^3 (2x^2 - 1) dx \\
 &= \left[\frac{2}{3}x^3 - x \right]_1^3 \\
 &= 18 - 3 - \frac{2}{3} + 1 \\
 &= \frac{46}{3}
 \end{aligned}$$

19. Find the length of the curve given by $x = \frac{1}{5}y^5 + \frac{1}{12y^3}$ between $(\frac{17}{60}, 1)$ and $(\frac{163}{480}, \frac{1}{2})$.

$$\begin{aligned}
 L &= \int ds = \int_{\frac{1}{2}}^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy & \frac{dx}{dy} &= y^4 - \frac{1}{4y^4} \\
 &= \int_{\frac{1}{2}}^1 \sqrt{1 + \left(y^4 - \frac{1}{4y^4}\right)^2} dy \\
 &= \int_{\frac{1}{2}}^1 \sqrt{1 + y^8 - \frac{1}{2} + \frac{1}{16y^8}} dy \\
 &= \int_{\frac{1}{2}}^1 \sqrt{y^8 + \frac{1}{2} + \frac{1}{16y^8}} dy \\
 &= \int_{\frac{1}{2}}^1 \sqrt{\left(y^4 + \frac{1}{4y^4}\right)^2} dy \\
 &= \int_{\frac{1}{2}}^1 \left(y^4 + \frac{1}{4y^4}\right) dy \\
 &= \left[\frac{1}{5}y^5 - \frac{1}{12y^3} \right]_{\frac{1}{2}}^1 \\
 &= \frac{1}{5} - \frac{1}{12} - \frac{1}{60} + \frac{8}{12} \\
 &= \frac{373}{480}
 \end{aligned}$$

20. Find the arc length function for the curve $y = \ln(x^2 - 1)$ starting at the point $P_0(2, \ln(3))$.

$$\begin{aligned}
 s(x) &= \int_a^x ds = \int_2^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dt & \frac{dx}{dy} &= \frac{2x}{x^2 - 1} \\
 &= \int_2^x \sqrt{1 + \frac{4t^2}{(t^2-1)^2}} dt \\
 &= \int_2^x \sqrt{\frac{t^4 - 2t^2 + 1 + 4t^2}{(t^2-1)^2}} dt \\
 &= \int_2^x \sqrt{\frac{t^4 + 2t^2 + 1}{(t^2-1)^2}} dt \\
 &= \int_2^x \frac{t^2 + 1}{t^2 - 1} dt \\
 &= \int_2^x \left(1 + \frac{2}{t^2 - 1}\right) dt \\
 &= \int_2^x \left(1 + \frac{1}{t-1} - \frac{1}{t+1}\right) dt \\
 &= \left[t + \ln|t-1| - \ln|t+1| \right]_2^x \\
 &= x + \ln|x-1| - \ln|x+1| - 2 + \ln 3
 \end{aligned}$$

21. The demand function for an Iphone is $p = D(x) = \frac{1000}{x+1}$ and the supply function is $p = S(x) = (x+1)^2$ where x in the number of units (in thousands) and p the price (in \$). Find the consumers' and producers' surplus at the equilibrium point.

$$\begin{aligned}
 \frac{1000}{x+1} &= (x+1)^2 & \bar{x} &= 9 & \text{Equilibrium point: } (9, 100) \\
 & & \bar{p} &= D(9) = 100 \\
 1000 &= (x+1)^3 \\
 10 &= x+1 \\
 x &= 9 \\
 CS &= \int_0^9 \left(\frac{1000}{x+1} - 100 \right) dx & PS &= \int_0^9 \left(100 - (x+1)^2 \right) dx \\
 &= \left[1000 \ln|x+1| - 100x \right]_0^9 & &= \left[100x - \frac{1}{3}(x+1)^2 \right]_0^9 \\
 &= 1000 \ln 10 - 900 & &= 900 - \frac{1000}{3} + \frac{1}{3} \\
 &\approx 1402.585 & &= 567
 \end{aligned}$$

Hence the consumer's surplus is \$1 402 585 and the producer's surplus \$567 000.

22. A company estimates that the revenue produced by a machine at time t will be $f(t) = 50 + 7t$ thousand dollars per year. Find the total revenue, the present value and the future value of the machine over the next 5 years, if money is worth 5% compounded continuously. Also, find the capital value.

$$\begin{aligned} \text{Total Revenue} &= \int_0^5 (50 + 7t) dt \\ &= \left[50t + \frac{7}{2}t^2 \right]_0^5 \\ &= 250 + \frac{175}{2} \\ &= \frac{675}{2} = \$337\,500 \end{aligned}$$

$$\begin{aligned} P &= \int_0^5 (50 + 7t)e^{-0.05t} dt & u &= 50 + 7t & v &= -20e^{-0.05t} \\ &= \left[-20(50 + 7t)e^{-0.05t} \right]_0^5 + \int_0^5 140e^{-0.05t} dt & du &= 7dt & dv &= e^{-0.05t} dt \\ &= -1700e^{-\frac{1}{4}} + 1000 - \left[2800e^{-0.05t} \right]_0^5 \\ &= -1700e^{-\frac{1}{4}} + 1000 - 2800e^{-\frac{1}{4}} + 2800 \\ &= 3800 - 4500e^{-\frac{1}{4}} \\ &\approx \$295\,396 \end{aligned}$$

$$\begin{aligned} A &= e^{(0.05)(5)} \int_0^5 (50 + 7t)e^{-0.05t} dt & u &= 50 + 7t & v &= -20e^{-0.05t} \\ &= e^{\frac{1}{4}} \left[-20(50 + 7t)e^{-0.05t} \right]_0^5 + e^{\frac{1}{4}} \int_0^5 140e^{-0.05t} dt & du &= 7dt & dv &= e^{-0.05t} dt \\ &= -1700 + 1000e^{\frac{1}{4}} - e^{\frac{1}{4}} \left[2800e^{-0.05t} \right]_0^5 \\ &= -1700 + 1000e^{\frac{1}{4}} - 2800 + 2800e^{\frac{1}{4}} \\ &= 3800e^{\frac{1}{4}} - 4500 \\ &\approx \$379\,297 \end{aligned}$$

$$\begin{aligned} CV &= \int_0^{\infty} (50 + 7t)e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b (50 + 7t)e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} \left(\left[-20(50 + 7t)e^{-0.05t} \right]_0^b + \int_0^b 140e^{-0.05t} dt \right) \\ &= \lim_{b \rightarrow \infty} \left(-1700e^{-\frac{b}{20}} + 1000 - \left[2800e^{-0.05t} \right]_0^b \right) \\ &= \lim_{b \rightarrow \infty} \left(-1700e^{-\frac{b}{20}} + 1000 - 2800e^{-\frac{b}{20}} + 2800 \right) \\ &= \lim_{b \rightarrow \infty} \left(3800 - 4500e^{-\frac{b}{20}} \right) & u &= 50 + 7t & v &= -20e^{-0.05t} \\ &= 3800 & du &= dt & dv &= e^{-0.05t} dt \\ &\approx \$3800\,000 \end{aligned}$$

23. Find the Gini index for the Lorenz curve $L(x) = \frac{4}{\pi} x \arctan(x)$.

$$\begin{aligned}
 G &= 2 \int_0^1 \left(x - \frac{4}{\pi} x \arctan x \right) dx & u &= \arctan x & v &= \frac{1}{2} x^2 \\
 &= \left[x^2 \right]_0^1 - \frac{8}{\pi} \int_0^1 x \arctan x dx & du &= \frac{1}{1+x^2} dx & dv &= x dx \\
 &= 1 - \frac{4}{\pi} \left[x^2 \arctan x \right]_0^1 + \frac{4}{\pi} \int_0^1 \frac{x^2}{x^2+1} dx & & & & \\
 &= 1 - \frac{4}{\pi} \frac{\pi}{4} + \frac{4}{\pi} \int_0^1 \left(1 - \frac{1}{x^2+1} \right) dx & & & & \\
 &= \frac{4}{\pi} \left[x - \arctan x \right]_0^1 & & & & \\
 &= \frac{4}{\pi} \left(1 - \frac{\pi}{4} \right) & & & & \\
 &= \frac{4}{\pi} - 1 & & & & \\
 &\approx 0.273 & & & &
 \end{aligned}$$

24. Solve the following differential equations.

a) $\frac{dy}{dx} = \frac{(x^2+1)(y^2+1)}{x^2-1}$

$$\begin{aligned}
 \frac{1}{y^2+1} dy &= \frac{x^2+1}{x^2-1} dx & & & & \\
 \int \frac{1}{y^2+1} dy &= \int \frac{x^2+1}{x^2-1} dx & & & & \\
 \arctan y &= \int \left(1 + \frac{2}{x^2-1} \right) dx & & & & \\
 \arctan y &= \int \left(1 + \frac{1}{x-1} - \frac{1}{x+1} \right) dx & & & & \\
 \arctan y &= x + \ln|x-1| - \ln|x+1| + C & & & &
 \end{aligned}$$

$$\begin{aligned}
 & & & & & \\
 & & & & & \\
 & & & & & \\
 \frac{2}{x^2-1} &= \frac{A}{x-1} + \frac{B}{x+1} & & & & \\
 2 &= A(x+1) + B(x-1) & & & & \\
 x=1: 2 &= 2A \Rightarrow A=1 & & & & \\
 x=-1: 2 &= -2B \Rightarrow B=-1 & & & &
 \end{aligned}$$

b) $\frac{dy}{dx} = \sec y \arctan x$

$$\begin{aligned}
 \cos y dy &= \arctan x dx & u &= \arctan x & v &= x \\
 \int \cos y dy &= \int \arctan x dx & du &= \frac{1}{1+x^2} dx & dv &= dx \\
 \sin y &= x \arctan x - \int \frac{x}{x^2+1} dx & & & & \\
 \sin y &= x \arctan x - \int \frac{1}{2} \frac{1}{u} du & & & & \\
 \sin y &= x \arctan x - \frac{1}{2} \ln|u| + C & & & & \\
 \sin y &= x \arctan x - \frac{1}{2} \ln|x^2+1| + C & & & &
 \end{aligned}$$

$$c) x^2 y dx + e^x \ln y dy = 0$$

$$e^x \ln y dy = -x^2 y dx$$

$$\frac{\ln y}{y} dy = -x^2 e^{-x} dx$$

$$\int \frac{\ln y}{y} dy = \int -x^2 e^{-x} dx$$

$$\int u du = x^2 e^{-x} - \int 2x e^{-x} dx$$

$$\frac{1}{2} u^2 = x^2 e^{-x} + 2x e^{-x} - \int 2e^{-x} dx$$

$$\frac{1}{2} (\ln |y|)^2 = x^2 e^{-x} + 2x e^{-x} + 2e^{-x} + C$$

$$\ln^2 |y| = 2x^2 e^{-x} + 4x e^{-x} + 4e^{-x} + D$$

$$u = \ln y$$

$$du = \frac{1}{y} dy$$

$$u = -x^2 \quad v = -e^{-x}$$

$$du = -2x dx \quad dv = e^{-x} dx$$

$$u = -2x \quad v = -e^{-x}$$

$$du = -2 dx \quad dv = e^{-x} dx$$

$$d) \frac{dy}{dx} = y \sin^2 x \quad y\left(\frac{\pi}{2}\right) = 2$$

$$\frac{1}{y} dy = \sin^2 x dx$$

$$\int \frac{1}{y} dy = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x\right) dx$$

$$\ln |y| = \frac{1}{2} x + \frac{1}{4} \sin 2x + C$$

$$y = B e^{\frac{1}{2}x + \frac{1}{4} \sin 2x}$$

$$\therefore y = 2e^{\frac{1}{2}x + \frac{1}{4} \sin 2x - \frac{\pi}{4}}$$

$$y\left(\frac{\pi}{2}\right) = 2$$

$$2 = B e^{\frac{\pi}{4}}$$

$$B = 2e^{-\frac{\pi}{4}}$$

25. If a new product becomes known both by word of mouth and by advertising, then p , the proportion of people who have heard of the new product after t weeks, satisfies a differential equation and initial condition of the form

$$\frac{dp}{dt} = 0.1(1-p) + 0.1p(1-p)$$

If initially 30% of the population have heard of the new product, what will be the proportion of have heard of it t weeks from now?

$$\frac{dp}{dt} = 0.1(1-p) + 0.1p(1-p)$$

$$\frac{dp}{dt} = 0.1(1-p)(1+p)$$

$$\frac{10}{(1-p)(1+p)} dp = dt$$

$$\int \frac{10}{(1-p)(1+p)} dp = \int dt$$

$$\int \left(\frac{5}{1-p} + \frac{5}{1+p} \right) dp = t + C$$

$$-5 \ln|1-p| + 5 \ln|1+p| = t + C$$

$$\ln \left| \frac{1+p}{1-p} \right| = \frac{1}{5}t + B$$

$$\frac{1+p}{1-p} = Ae^{\frac{1}{5}t}$$

Since $p(0) = 0.3$, then $\frac{1+0.3}{1-0.3} = Ae^0$, that is, $A = \frac{13}{7}$.

Solving for p ,

$$\frac{1+p}{1-p} = \frac{13}{7}e^{\frac{1}{5}t}$$

$$7 + 7p = 13e^{\frac{1}{5}t} - 13pe^{\frac{1}{5}t}$$

$$p(7 + 13e^{\frac{1}{5}t}) = 13e^{\frac{1}{5}t} - 7$$

$$p(t) = \frac{13e^{\frac{1}{5}t} - 7}{13e^{\frac{1}{5}t} + 7}$$

26. Determine if the following sequences are monotonic, bounded and convergent.

a) $a_n = \frac{\sqrt{n}}{3^n + 1}$

$$f(x) = \frac{\sqrt{x}}{3^x + 1} \quad f'(x) = \frac{\frac{1}{2\sqrt{x}}(3^x + 1) - \sqrt{x}3^x \ln 3}{(3^x + 1)^2}$$

$$= \frac{3^x + 1 - 2x3^x \ln 3}{2\sqrt{x}(3^x + 1)^2}$$

$$= \frac{3^x \left(1 + \frac{1}{3^x} - 2x \ln 3\right)}{2\sqrt{x}(3^x + 1)^2} < 0 \quad \text{if } x \geq 1$$

Thus a_n is decreasing.

It is bounded: $0 \leq a_n \leq \frac{1}{4}$

It converges: $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{3^n + 1} \underset{\text{HR } \frac{\infty}{\infty}}{=} \lim_{n \rightarrow \infty} \frac{\frac{1}{2\sqrt{n}}}{3^n \ln 3} = \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}3^n \ln 3} = 0$

b) $\left\langle 5 - \left(-\frac{2}{3}\right)^n \right\rangle$

$$a_n : 5 + \frac{2}{3}, 5 - \frac{4}{9}, 5 + \frac{8}{27}, \dots$$

a_n is nonmonotonic (alternates between >5 and <5)

$$\text{Since } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(5 - \left(-\frac{2}{3}\right)^n\right) = 5$$

$$\text{then it is bounded: } 5 - \frac{4}{9} \leq a_n \leq 5 + \frac{2}{3}$$

$$\frac{41}{9} \leq a_n \leq \frac{17}{3}$$

$$\text{It converges: } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(5 - \left(-\frac{2}{3}\right)^n\right) = 5$$

c) $\left\langle \frac{(-1)^n}{n!} \right\rangle$

a_n is nonmonotonic (alternates between + and -)

$$\text{It is bounded: } -1 \leq a_n \leq \frac{1}{2}$$

$$\text{It converges: } \lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{n!} = 0 \quad \text{thus } \lim_{n \rightarrow \infty} a_n = 0$$

d) $a_n = \frac{n}{\ln(n+1)}$

$$f(x) = \frac{x}{\ln(x+1)} \quad f'(x) = \frac{\ln(x+1) - x \frac{1}{x}}{(\ln(x+1))^2} = \frac{\ln(x+1) - 1}{(\ln(x+1))^2} > 0 \quad \text{for } x \geq e-1$$

Thus a_n is increasing (for $n \geq 2$)

It is only bounded below: $0 \leq a_n$

$$\text{It diverges: } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{\ln(n+1)} \stackrel{\text{HR } \frac{\infty}{\infty}}{=} \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n+1}} = \lim_{n \rightarrow \infty} (n+1) = \infty$$

27. Determine if the following series converge. If so, find the sum.

$$\begin{aligned} \text{a) } \sum_{n=1}^{\infty} \frac{4^{2n+1}}{3^{3n-1}} &= \frac{64}{9} + \frac{1024}{243} + \dots \\ &= \sum_{n=1}^{\infty} \frac{64}{9} \left(\frac{16}{27}\right)^{n-1} \quad \text{geometric series with } |r| = \frac{16}{27} < 1 \\ &= \frac{\frac{64}{9}}{1 - \frac{16}{27}} \\ &= \frac{192}{11} \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \dots &= \sum_{n=1}^{\infty} \frac{1}{3} \left(-\frac{2}{3}\right)^{n-1} && \text{geometric series with } |r| = \frac{2}{3} < 1 \\ &= \frac{\frac{1}{3}}{1 + \frac{2}{3}} \\ &= \frac{1}{5} \end{aligned}$$

$$\text{c) } \sum_{n=1}^{\infty} \frac{2}{n(n+2)}$$

$$\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{A(n+2) + Bn}{n(n+2)}$$

$$2 = A(n+2) + Bn$$

$$n=0 \quad 2 = 2A \Rightarrow A = 1$$

$$n=-2 \quad 2 = -2B \Rightarrow B = -1$$

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) \quad (\text{telescopic series})$$

$$\begin{aligned} s_n &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \end{aligned}$$

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}$$

$$\text{Hence } \sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \frac{3}{2}$$

$$\text{d) } \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{n!} = 3 \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} = 3e^{-3} \quad \text{since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\text{e) } \frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} = \sin \frac{\pi}{2} = 1 \quad \text{since } \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\text{f) } \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)(4n+1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)(3n+1)} \cdot \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{1 \cdot 4 \cdot 7 \cdots (3n-2)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(4n+1) \frac{1}{n}}{(3n+1) \frac{1}{n}} \right| = \lim_{n \rightarrow \infty} \frac{4 + \frac{1}{n}}{3 + \frac{1}{n}} = \frac{4}{3} > 1$$

$$\text{Hence } \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{1 \cdot 4 \cdot 7 \cdots (3n-2)} \text{ diverges by the ratio test}$$

28. Determine whether the following series are convergent or divergent.

$$\text{a) } \sum_{n=1}^{\infty} \frac{1}{2\sqrt[3]{n}-1} \quad \text{Let } b_n = \frac{1}{n^{\frac{1}{3}}}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2n^{\frac{1}{3}}-1}}{\frac{1}{n^{\frac{1}{3}}}} = \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{3}}}{2n^{\frac{1}{3}}-1} \cdot \frac{1}{n^{\frac{1}{3}}} = \lim_{n \rightarrow \infty} \frac{1}{2-\frac{1}{n^{\frac{1}{3}}}} = \frac{1}{2}$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ diverges (p -series with $p = \frac{1}{3} \leq 1$) then $\sum_{n=1}^{\infty} \frac{1}{2\sqrt[3]{n}-1}$ diverges by the limit comparison test.

$$\text{b) } \sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{n(1+n^2)} \quad \text{Let } b_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{\frac{5}{2}}}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{2\sqrt{n}+1}{n+n^3}}{\frac{1}{n^{\frac{5}{2}}}} = \lim_{n \rightarrow \infty} \frac{(2\sqrt{n}+1)n^{\frac{5}{2}}}{n+n^3} = \lim_{n \rightarrow \infty} \frac{2n^3+n^{\frac{5}{2}}}{n^3+n} \cdot \frac{1}{n^{\frac{1}{2}}} = \lim_{n \rightarrow \infty} \frac{2+\frac{1}{n^{\frac{1}{2}}}}{1+\frac{1}{n^2}} = 2$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$ converges (p -series with $p = \frac{5}{2} > 1$) then $\sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{n(1+n^2)}$ converges by the limit comparison test.

$$\text{c) } \sum_{n=2}^{\infty} \frac{e^{\sqrt{n}}}{4\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \frac{e^{\sqrt{n}}}{4\sqrt{n}} \underset{\text{HR } \frac{\infty}{\infty}}{=} \lim_{n \rightarrow \infty} \frac{e^{\sqrt{n}} \cdot \frac{1}{2\sqrt{n}}}{4 \cdot \frac{1}{2\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{1}{4} e^{\sqrt{n}} = \infty$$

Thus $\sum_{n=2}^{\infty} \frac{e^{\sqrt{n}}}{4\sqrt{n}}$ diverges by the divergence test.

$$\text{d) } \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1} \quad b_n = \frac{\sqrt{n}}{n+1}$$

$$\text{(i) } f(x) = \frac{\sqrt{x}}{x+1} \quad f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+1) - \sqrt{x}}{(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} < 0 \quad \text{for } x > 1$$

Thus b_n is decreasing

$$\text{(ii) } \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} \cdot \frac{1}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} + \frac{1}{\sqrt{n}}} = 0$$

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$ converges by the alternating series test.

$$\text{e) } \sum_{n=1}^{\infty} \frac{n^3}{3^{n+1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^3}{3^{n+2}}}{\frac{n^3}{3^{n+1}}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3 3^{n+1}}{3^{n+2} n^3} \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^3 \frac{1}{3} \right| = \lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n} \right)^3 \frac{1}{3} \right| = \frac{1}{3} < 1$$

Thus $\sum_{n=1}^{\infty} \frac{k^3}{3^{k+1}}$ converges by the Ratio Test.

$$f) \sum_{n=1}^{\infty} \frac{n+2}{(n+1)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{n+3}{(n+2)!}}{\frac{n+2}{(n+1)!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+3)(n+1)!}{(n+2)(n+1)!(n+2)} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+3}{(n+2)^2} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n+3}{n^2+4n+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{4}{n} + \frac{4}{n^2}} \right| = 0 < 1 \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)!}$ converges by the Ratio Test.

$$g) \sum_{n=1}^{\infty} \left(\frac{3}{3+5n}\right)^{2n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{3}{3+5n}\right)^{2n}} = \lim_{n \rightarrow \infty} \left(\frac{3}{3+5n}\right)^2 = 0 < 1$$

Thus $\sum_{n=1}^{\infty} \left(\frac{3}{3+5n}\right)^{2n}$ converges by the root test.

$$h) \sum_{n=0}^{\infty} \arctan n$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2}$$

Thus $\sum_{n=0}^{\infty} \arctan n$ diverges by the divergence test.

29. Determine whether the series converges absolutely, converges conditionally or diverges.

$$a) \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{n} \quad \text{Let } f(x) = \frac{\ln x}{x}$$

$$1. \quad f(x) \geq 0 \quad \text{on } [1, \infty)$$

$$2. \quad f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0 \quad \text{for } x \geq 3$$

thus f is decreasing on $[3, \infty)$.

$$3. \quad f \text{ is continuous on } [1, \infty) \text{ (quotient of two continuous functions, logarithmic and polynomial)}$$

$$\begin{aligned}
 \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx && u = \ln x \\
 &= \lim_{t \rightarrow \infty} \int_{x=1}^{x=t} u du && du = \frac{1}{x} dx \\
 &= \lim_{t \rightarrow \infty} \left[\frac{1}{2} u^2 \right]_{x=1}^{x=t} \\
 &= \lim_{t \rightarrow \infty} \left[\frac{1}{2} (\ln x)^2 \right]_1^t \\
 &= \lim_{t \rightarrow \infty} \frac{1}{2} (\ln t)^2 \\
 &= \infty
 \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the integral test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n} \quad b_n = \frac{n+1}{n^2}$$

$$(i) f(x) = \frac{\ln x}{x} \quad f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0 \quad \text{for } x \geq 3$$

Hence b_n is decreasing

$$(ii) \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} \underset{\substack{\uparrow \\ HR \frac{\infty}{\infty}}}{=} \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = 0$$

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ converges by the alternating series test.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ converges conditionally.

$$b) \sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3 + 4}}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{\sqrt{n^3 + 4}} \right| = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 4}} \quad \text{Let } b_n = \frac{n}{\sqrt{n^3}} = \frac{1}{n^{\frac{1}{2}}}$$

$$\text{then } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n}{\sqrt{n^3 + 4}}}{\frac{1}{n^{\frac{1}{2}}}} = \lim_{n \rightarrow \infty} \frac{n^{\frac{3}{2}}}{\sqrt{n^3 \left(1 + \frac{4}{n^3}\right)}} = \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{1 + \frac{4}{n^3}}} \right) = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges (p -series with $p = \frac{1}{2} \leq 1$), then $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 4}}$ diverges by the limit comparison test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+4}} \quad b_n = \frac{n}{\sqrt{n^3+4}}$$

$$(i) f(x) = \frac{x}{\sqrt{x^3+4}}$$

$$f'(x) = \frac{\sqrt{x^3+4} - x \frac{1}{2\sqrt{x^3+4}} 3x^2}{x^3+4} = \frac{8-x^3}{(x^3+4)^{\frac{3}{2}}} < 0 \text{ for } x > 2$$

Hence b_n is decreasing

$$(ii) \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^3+4}} = \lim_{n \rightarrow \infty} \frac{n}{n^{\frac{3}{2}} \sqrt{1+\frac{4}{n^3}}} = \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{1}{2}} \sqrt{1+\frac{4}{n^3}}} = 0$$

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+4}}$ converges by the alternating series test.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+4}}$ converges conditionally.

30. Find the radius and interval of convergence for the following power series.

$$a) \sum_{n=1}^{\infty} \frac{(n-1)x^n}{n2^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{nx^{n+1}}{(n+1)2^{n+1}}}{\frac{(n-1)x^n}{n2^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^2 x^{n+1} 2^n}{(n+1)(n-1)x^n 2^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{xn^2}{2n^2-2} \cdot \frac{1}{\frac{1}{n^2}} \right| = \left| \frac{x}{2-\frac{2}{n^2}} \right| = \left| \frac{x}{2} \right|$$

By the ratio test, the power series converges for $\left| \frac{x}{2} \right| < 1$

$$\begin{aligned} |x| &< 2 \quad \therefore R = 2 \\ -2 &< x < 2 \end{aligned}$$

$$\text{At } x = -2, \sum_{n=1}^{\infty} \frac{(n-1)(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (n-1)}{n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n (n-1)}{n} \right| = \lim_{n \rightarrow \infty} \left| 1 - \frac{1}{n} \right| = 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{(-1)^n (n-1)}{n} \not\rightarrow 0$$

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n (n-1)}{n}$ diverges by the divergence test.

$$\text{At } x = 2, \sum_{n=1}^{\infty} \frac{(n-1)2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{n-1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{n-1}{n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 1$$

Hence $\sum_{n=1}^{\infty} \frac{n-1}{n}$ diverges by the divergence test.

Hence $I = (-2, 2)$ and $R = 2$.

$$\text{b) } \sum_{n=1}^{\infty} \frac{(3x-4)^n}{5^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(3x-4)^{n+1}}{5^{n+1}}}{\frac{(3x-4)^n}{5^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x-4)^{n+1} 5^n}{5^{n+1} (3x-4)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3x-4}{5} \right| = \left| \frac{3x-4}{5} \right|$$

By the ratio test, the power series converges for $\left| \frac{3x-4}{5} \right| < 1$

$$|3x-4| < 5$$

$$-5 < 3x-4 < 5$$

$$-1 < 3x < 9$$

$$-\frac{1}{3} < x < 3$$

$$\text{At } x = \frac{-1}{3}, \sum_{n=1}^{\infty} \frac{(3(\frac{-1}{3})-4)^n}{5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n$$

$$\lim_{n \rightarrow \infty} (-1)^n \not\exists$$

Hence $\sum_{n=1}^{\infty} (-1)^n$ diverges by the divergence test.

$$\text{At } x = 3, \sum_{n=1}^{\infty} \frac{(3 \cdot 3 - 4)^n}{5^n} = \sum_{n=1}^{\infty} \frac{5^n}{5^n} = \sum_{n=1}^{\infty} 1$$

$$\lim_{n \rightarrow \infty} 1 = 1$$

Hence $\sum_{n=1}^{\infty} 1$ diverges by the divergence test.

Hence $I = (-\frac{1}{3}, 3)$ and $R = \frac{5}{3}$.

$$c) \sum_{n=1}^{\infty} \frac{(7-2x)^n}{n(n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(7-2x)^{n+1}}{(n+1)(n+2)}}{\frac{(7-2x)^n}{n(n+1)}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(7-2x)^{n+1} (n^2 + n)}{(n^2 + 3n + 2)(7-2x)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(7-2x)(1 + \frac{1}{n})}{1 + \frac{3}{n} + \frac{2}{n^2}} \right| = |7-2x|$$

By the ratio test, the power series converges for $|7-2x| < 1$

$$-1 < 7-2x < 1$$

$$-1 < 2x-7 < 1$$

$$6 < 2x < 8$$

$$3 < x < 4$$

$$\text{At } x=3, \sum_{n=1}^{\infty} \frac{(7-2 \cdot 3)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+n} \quad \text{Let } b_n = \frac{1}{n^2}$$

$$\text{then } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2+n}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+n} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p -series with $p=2 > 1$), then $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges

by the limit comparison test.

$$\text{At } x=4, \sum_{n=1}^{\infty} \frac{(7-2 \cdot 4)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$$

$$\text{Since } \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2+n} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2+n} \text{ converges (see above) then } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$$

converges absolutely, thus converges.

Hence $I = [3, 4]$ and $R = \frac{1}{2}$.

$$d) \sum_{n=1}^{\infty} \frac{(x+4)^n}{\sqrt{n}}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(x+4)^{n+1}}{\sqrt{n+1}}}{\frac{(x+4)^n}{\sqrt{n}}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+4)^{n+1} \sqrt{n}}{\sqrt{n+1} (x+4)^n} \right| = \lim_{n \rightarrow \infty} \left| (x+4) \frac{\sqrt{n}}{\sqrt{n+1} \sqrt{1+\frac{1}{n}}} \right| \\ &= \lim_{n \rightarrow \infty} \left| (x+4) \frac{1}{\sqrt{1+\frac{1}{n}}} \right| = |x+4| \end{aligned}$$

By the ratio test, the power series converges for $|x+4| < 1$
 $-1 < x+4 < 1$
 $-5 < x < -3$

$$\text{At } x = -5, \sum_{n=1}^{\infty} \frac{(-5+4)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{Let } b_n = \frac{1}{\sqrt{n}}$$

$$(i) f(x) = \frac{1}{\sqrt{x}} \quad f'(x) = \frac{-1}{2x^{3/2}} < 0 \quad \text{for } x \geq 1$$

Hence b_n is decreasing

$$(ii) \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test.

$$\text{At } x = -3, \sum_{n=1}^{\infty} \frac{(-3+4)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{which diverges (} p\text{-series with } p = \frac{1}{2} \leq 1)$$

Hence $I = [-5, -3)$ and $R = 1$.

$$e) \sum_{n=1}^{\infty} \frac{5^n (2x-1)^n}{3 \cdot 10 \cdot 17 \cdots (7n-4)}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{5^{n+1} (2x-1)^{n+1}}{3 \cdot 10 \cdot 17 \cdots (7n-4)(7n+3)}}{\frac{5^n (2x-1)^n}{3 \cdot 10 \cdot 17 \cdots (7n-4)}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{5^{n+1} (2x-1)^{n+1} \cdot 3 \cdot 10 \cdot 17 \cdots (7n-4)}{5^n (2x-1)^n \cdot 3 \cdot 10 \cdot 17 \cdots (7n-4)(7n+3)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{5(2x-1)}{7n+3} \right| = 0 \end{aligned}$$

By the ratio test, the power series converges for all values of x .
 Ergo, $R = \infty$ and $I = \mathbb{R}$

$$f) \sum_{n=1}^{\infty} \frac{(3n)!(x + \frac{1}{2})^n}{3^n (2n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(3n+3)!(x + \frac{1}{2})^{n+1}}{3^{n+1}(2n+2)!}}{\frac{(3n)!(x + \frac{1}{2})^n}{3^n (2n)!}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(3n+3)!(x + \frac{1}{2})^{n+1} 3^n (2n)!}{(3n)!(x + \frac{1}{2})^n 3^{n+1} (2n+2)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(3n+3)(3n+2)(3n+1)(3n)!(x + \frac{1}{2})(2n)!}{(3n)!(2n+2)(2n+1)(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3(3n+2)(3n+1)(x + \frac{1}{2})}{2(2n+1)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(27n^2 + 27n + 6)(x + \frac{1}{2}) \frac{1}{n}}{4n + 2} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(27n + 27 + \frac{6}{n})(x + \frac{1}{2})}{4 + \frac{2}{n}} \right| \\ &= \begin{cases} \infty & \text{if } x \neq \frac{-1}{2} \\ 0 & \text{if } x = \frac{-1}{2} \end{cases} \end{aligned}$$

By the ration test, the power series converges only when $x = \frac{-1}{2}$.

Ergo, $R = 0$ and $I = \{\frac{-1}{2}\}$

31. Find the Maclaurin Series for $f(x)$ using the definition.

$$\begin{aligned} \text{a) } f(x) &= \frac{1}{1+2x} = (1+2x)^{-1} & f(0) &= 1 \\ f'(x) &= -2(1+2x)^{-2} & f'(0) &= -2 \\ f''(x) &= 2 \cdot 2^2 (1+2x)^{-3} & f''(0) &= 2 \cdot 2^2 \\ f'''(x) &= -3 \cdot 2 \cdot 2^3 (1+2x)^{-4} & f'''(0) &= -3 \cdot 2 \cdot 2^3 \\ f^{iv}(x) &= 4 \cdot 3 \cdot 2 \cdot 2^4 (1+2x)^{-5} & f^{iv}(0) &= 4 \cdot 3 \cdot 2 \cdot 2^4 \\ & \vdots & & \vdots \\ f^{(n)}(x) &= (-1)^n n! 2^n (1+2x)^{-n-1} & f^{(n)}(0) &= (-1)^n n! 2^n \\ f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n n! 2^n x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n 2^n x^n \end{aligned}$$

$$\begin{aligned}
 \text{b) } f(x) &= \cos(4x) & f(0) &= 1 \\
 f'(x) &= -4 \sin 4x & f'(0) &= 0 \\
 f''(x) &= -16 \cos 4x & f''(0) &= -4^2 = -16 \\
 f'''(x) &= 64 \sin 4x & f'''(0) &= 0 \\
 f^{iv}(x) &= 256 \cos 4x & f^{iv}(0) &= 4^4 = 16^2 \\
 & \vdots & & \vdots \\
 f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 - \frac{4^2 x^2}{2!} + \frac{4^4 x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 16^n x^{2n}}{(2n)!}
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } f(x) &= \sqrt[3]{x+8} = (x+8)^{\frac{1}{3}} & f(0) &= 2 \\
 f'(x) &= \frac{1}{3}(x+8)^{-\frac{2}{3}} & f'(x) &= \frac{1}{3 \cdot 2^2} \\
 f''(x) &= \frac{-2}{3^2}(x+8)^{-\frac{5}{3}} & f''(x) &= \frac{-2}{3^2 \cdot 2^5} \\
 f'''(x) &= \frac{2 \cdot 5}{3^3}(x+8)^{-\frac{8}{3}} & f'''(x) &= \frac{2 \cdot 5}{3^3 \cdot 2^8} \\
 f^{iv}(x) &= \frac{-2 \cdot 5 \cdot 8}{3^4}(x+8)^{-\frac{11}{3}} & f^{iv}(x) &= \frac{-2 \cdot 5 \cdot 8}{3^4 \cdot 2^{11}} \\
 & \vdots & & \vdots \\
 f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 2 + \frac{x}{3 \cdot 2^2 \cdot 1!} - \frac{2x^2}{3^2 \cdot 2^5 \cdot 2!} + \frac{2 \cdot 5x^3}{3^3 \cdot 2^8 \cdot 3!} + \frac{2 \cdot 5 \cdot 8x^4}{3^4 \cdot 2^{11} \cdot 4!} \dots \\
 &= 2 + \frac{x}{12} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2 \cdot 5 \cdot 8 \dots (3n-4)x^n}{3^n 2^{3n-1} n!}
 \end{aligned}$$

32. Find the Taylor series for $f(x)$ at $x = a$.

$$\begin{aligned}
 \text{a) } f(x) &= \sin 3x \quad a = \frac{\pi}{3} & f\left(\frac{\pi}{3}\right) &= 0 \\
 f'(x) &= 3 \cos 3x & f'\left(\frac{\pi}{3}\right) &= -3 \\
 f''(x) &= -3^2 \sin 3x & f''\left(\frac{\pi}{3}\right) &= 0 \\
 f'''(x) &= -3^3 \cos 3x & f'''\left(\frac{\pi}{3}\right) &= 3^3 \\
 f^{iv}(x) &= 3^4 \sin 3x & f^{iv}\left(\frac{\pi}{3}\right) &= 0 \\
 & \vdots & & \vdots \\
 f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)(x-a)^n}{n!} = -\frac{3(x-\frac{\pi}{3})}{1!} + \frac{3^3(x-\frac{\pi}{3})^3}{3!} - \frac{3^5(x-\frac{\pi}{3})^5}{5!} \dots \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{2n+1} (x-\frac{\pi}{3})^{2n+1}}{(2n+1)!}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } f(x) &= \frac{1}{(6-5x)^2} = (6-5x)^{-2} & a &= 1 & f(1) &= 1 \\
 f'(x) &= 2 \cdot 5(6-5x)^{-3} & f'(1) &= 2 \cdot 5 \\
 f''(x) &= 3 \cdot 2 \cdot 5^2(6-5x)^{-4} & f''(1) &= 3 \cdot 2 \cdot 5^2 \\
 f'''(x) &= 4 \cdot 3 \cdot 2 \cdot 5^3(6-5x)^{-5} & f'''(1) &= 4 \cdot 3 \cdot 2 \cdot 5^3 \\
 f^{iv}(x) &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5^4(6-5x)^{-6} & f^{iv}(1) &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5^4 \\
 & \vdots & & \vdots \\
 f^{(n)}(x) &= (n+1)!5^n(6-5x)^{-n-2} & f^{(n)}(1) &= (n+1)!5^n \\
 f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)!5^n(x-1)^n}{n!} = \sum_{n=0}^{\infty} (n+1)5^n(x-1)^n
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } f(x) &= \sqrt[5]{x} = x^{\frac{1}{5}} & a &= 32 & f(32) &= 2 \\
 f'(x) &= \frac{1}{5}x^{-\frac{4}{5}} & f'(32) &= \frac{1}{5 \cdot 2^4} \\
 f''(x) &= \frac{4}{5^2}x^{-\frac{9}{5}} & f''(32) &= \frac{-4}{5^2 \cdot 2^9} \\
 f'''(x) &= \frac{4 \cdot 9}{5^3}x^{-\frac{14}{5}} & f'''(32) &= \frac{4 \cdot 9}{5^3 \cdot 2^{14}} \\
 f^{iv}(x) &= \frac{-4 \cdot 9 \cdot 14}{5^4}x^{-\frac{19}{5}} & f^{iv}(32) &= \frac{-4 \cdot 9 \cdot 14}{5^4 \cdot 2^{19}} \\
 & \vdots & & \vdots \\
 f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(32)(x-32)^n}{n!} = f(32) + \frac{f'(32)(x-32)}{1!} + \frac{f''(32)(x-32)^2}{2!} + \dots \\
 &= 2 + \frac{x-32}{80} - \frac{4(x-32)^2}{5^2 2^9 2!} + \frac{4 \cdot 9(x-32)^3}{5^3 2^{14} 3!} - \frac{4 \cdot 9 \cdot 14(x-32)^4}{5^4 2^{19} 4!} + \dots \\
 &= 2 + \frac{x-32}{80} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)(x-32)^n}{5^n 2^{5n-1} n!}
 \end{aligned}$$

33. Use known Maclaurin series to obtain the Maclaurin series for the given function. Give the radius of convergence.

$$\text{a) } f(x) = \sin 3x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad R = \infty$$

$$\sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!} \quad R = \infty$$

$$\text{b) } f(x) = \frac{1 - \cos 2x}{x}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad R = \infty$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \dots$$

$$1 - \cos 2x = \frac{2^2 x^2}{2!} - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n} x^{2n}}{(2n)!}$$

$$\frac{1 - \cos 2x}{x} = \frac{2^2 x^1}{2!} - \frac{2^4 x^3}{4!} + \frac{2^6 x^5}{6!} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n} x^{2n-1}}{(2n)!} \quad R = \infty$$

$$\text{c) } f(x) = \frac{x}{1+x^2}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \quad R = 1$$

$$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\frac{x}{1+x^2} = x - x^3 + x^5 - x^7 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \quad R = 1$$

34. Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

$$\text{a) } f(x) = \sin x \cos x$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)$$

$$= x - \frac{x^3}{3!} - \frac{x^3}{2!} + \frac{x^5}{5!} + \frac{x^5}{3!2!} + \frac{x^5}{4!} - \dots$$

$$= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \dots$$

$$\begin{aligned}
 \text{b) } f(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
 e^x - e^{-x} &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\right) \\
 &= 2x + \frac{1}{3}x^3 + \frac{1}{60}x^5 + \dots \\
 e^x + e^{-x} &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\right) \\
 &= 2 + x^2 + \frac{1}{12}x^4 + \dots \\
 &\frac{x - \frac{1}{3}x^3 + \frac{2}{15}x^5}{2 + x^2 + \frac{1}{12}x^4 + \dots} \frac{2x + \frac{1}{3}x^3 + \frac{1}{60}x^5 + \dots}{2x + \frac{1}{3}x^3 + \frac{1}{60}x^5 + \dots} \\
 &\frac{2x + x^3 + \frac{1}{12}x^5 + \dots}{-\frac{2}{3}x^3 - \frac{1}{15}x^5 + \dots} \\
 &\frac{-\frac{2}{3}x^3 - \frac{1}{3}x^5 + \dots}{\frac{4}{15}x^5 + \dots} \\
 &\frac{\frac{4}{15}x^5 + \dots}{\dots} \\
 \frac{e^x - e^{-x}}{e^x + e^{-x}} &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \dots
 \end{aligned}$$

35. Approximate f by a Taylor polynomial with degree n at the number a . Use this approximation to estimate the given number, and estimate the accuracy of that approximation.

$$\text{a) } f(x) = \sqrt{1+2x} \quad n=3 \quad a=4 \quad \sqrt{9.2}$$

$$\begin{aligned}
 f(x) &= (1+2x)^{\frac{1}{2}} & f(4) &= 3 \\
 f'(x) &= (1+2x)^{-\frac{1}{2}} & f'(4) &= \frac{1}{3} \\
 f''(x) &= -(1+2x)^{-\frac{3}{2}} & f''(4) &= -\frac{1}{27} \\
 f'''(x) &= 3(1+2x)^{-\frac{5}{2}} & f'''(4) &= \frac{1}{81}
 \end{aligned}$$

$$\begin{aligned}
 T_3(x) &= f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3 \\
 &= 3 + \frac{1}{3}(x-4) - \frac{1}{54}(x-4)^2 + \frac{1}{486}(x-4)^3
 \end{aligned}$$

$$\text{Ergo, } \sqrt{1+2x} \approx 3 + \frac{1}{3}(x-4) - \frac{1}{54}(x-4)^2 + \frac{1}{486}(x-4)^3$$

$$\text{At } x = 4.1, \sqrt{9.2} \approx 3 + \frac{1}{30} - \frac{1}{5400} + \frac{1}{486000} \approx 3.0331502$$

$$\text{Since } f^{(iv)}(x) = \frac{-15}{(1+2x)^{\frac{7}{2}}}, \text{ then } f^{(iv)}(x) = \frac{-15}{(1+2x)^{\frac{7}{2}}} \leq \frac{15}{(1+2 \cdot 3.9)^{\frac{7}{2}}} < 0.0074 \text{ on } |x-4| \leq \frac{1}{10},$$

$$\text{Ergo } |R_3(x)| \leq \frac{M}{4!} |x-4|^4 < \frac{0.0074}{4!} \left(\frac{1}{10}\right)^4 < 0.000000031 \text{ on } |x-4| \leq \frac{1}{10}.$$

$$\text{b) } f(x) = \sec x \quad n = 2 \quad a = \frac{\pi}{4} \quad \sec(43^\circ)$$

$$f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \quad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$T_2(x) = f\left(\frac{\pi}{4}\right) + \frac{f'\left(\frac{\pi}{4}\right)}{1!} \left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!} \left(x - \frac{\pi}{4}\right)^2$$

$$= \sqrt{2} + \sqrt{2} \left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right)^2$$

$$\text{Ergo, } \sec x \approx \sqrt{2} + \sqrt{2} \left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right)^2$$

$$\text{At } x = 43 \frac{\pi}{180} = \frac{43\pi}{180}, \quad \sec 43^\circ = \sec \frac{43\pi}{180} \approx \sqrt{2} - \sqrt{2} \frac{\pi}{90} + \frac{3\sqrt{2}}{2} \left(\frac{\pi}{90}\right)^2 \approx 1.36743$$

$$\text{Since } f'''(x) = \sec \tan^3 x + 5 \sec^3 \tan x,$$

$$\text{then } f'''(x) = \sec x \tan^3 x + 5 \sec^3 x \tan x$$

$$\leq \sec \frac{47\pi}{180} \tan^3 \frac{47\pi}{180} + 5 \sec^3 \frac{47\pi}{180} \tan \frac{47\pi}{180} < 18.8 \quad \text{on } \left|x - \frac{\pi}{4}\right| < \frac{\pi}{90}$$

$$\text{Ergo } |R_2(x)| \leq \frac{M}{3!} |x - \frac{\pi}{4}|^3 < \frac{18.8}{3!} \left(\frac{\pi}{90}\right)^3 < 0.00013 \text{ on } \left|x - \frac{\pi}{4}\right| < \frac{\pi}{90}.$$

36. Use series to approximate the definite integral to within five decimal places.

$$\text{a) } \int_0^{\frac{1}{2}} x \sin(x^3) dx$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$$

$$x \sin x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}$$

$$\int_0^{\frac{1}{2}} x \sin(x^3) dx = \int_0^{\frac{1}{2}} \left(x^4 - \frac{1}{6} x^{10} + \frac{1}{120} x^{16} - \dots\right) dx$$

$$= \left[\frac{1}{5} x^5 - \frac{1}{66} x^{11} + \frac{1}{2040} x^{17} - \dots\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{160} - \frac{1}{135168} + \frac{1}{534773760} - \dots$$

$$\text{Since } \frac{1}{534773760} \approx 0.0000000019 < 0.000005, \text{ then}$$

$$\int_0^{\frac{1}{2}} x \sin(x^3) dx \approx \frac{1}{160} - \frac{1}{135168} \approx 0.00624$$

$$\text{b) } \int_0^{\frac{1}{5}} e^{-\frac{x^2}{2}} dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{-x^2}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{2^2 2!} - \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} - \dots$$

$$\int_0^{\frac{1}{5}} e^{-\frac{x^2}{2}} dx = \int_0^{\frac{1}{5}} \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \dots\right) dx$$

$$= \left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \dots \right]_0^{\frac{1}{5}}$$

$$= \frac{1}{5} - \frac{1}{750} + \frac{1}{125000} - \frac{1}{26250000} + \dots$$

Since $\frac{1}{26250000} \approx 0.00000004 < 0.000005$, then

$$\int_0^{\frac{1}{5}} e^{-\frac{x^2}{2}} dx \approx \frac{1}{5} - \frac{1}{750} + \frac{1}{125000} = \frac{74503}{375000} \approx 0.19867$$

$$\text{c) } \int_0^{\frac{1}{3}} \frac{1}{1+x^4} dx$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(-x^4)} = 1 + (-x^4) + (-x^4)^2 + (-x^4)^3 + \dots = \sum_{n=0}^{\infty} (-x^4)^n$$

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + \dots = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

$$\int_0^{\frac{1}{3}} \frac{1}{1+x^4} dx = \int_0^{\frac{1}{3}} (1 - x^4 + x^8 - x^{12} + \dots) dx$$

$$= \left[x - \frac{1}{5} x^5 + \frac{1}{9} x^9 - \frac{1}{13} x^{13} + \dots \right]_0^{\frac{1}{3}}$$

$$= \frac{1}{3} - \frac{1}{1215} + \frac{1}{177147} - \frac{1}{20726199} + \dots$$

Since $\frac{1}{20726199} \approx 0.00000005 < 0.000005$, then

$$\int_0^{\frac{1}{3}} \frac{1}{1+x^4} dx \approx \frac{1}{3} - \frac{1}{1215} + \frac{1}{177147} = \frac{294521}{885735} \approx 0.33252$$

37. Consider the function $f(x) = \ln(2x+1)$.

a) Find the Maclaurin series for $f(x)$ using the definition.

$$f(x) = \ln(2x+1) \qquad f(0) = 0$$

$$f'(x) = \frac{2}{2x+1} = 2(2x+1)^{-1} \qquad f'(0) = 2$$

$$f''(x) = -2^2(2x+1)^{-2} \qquad f''(0) = -2^2$$

$$f'''(x) = 2^3(2)(2x+1)^{-3} \qquad f'''(0) = 2^3(2)$$

$$f^{iv}(x) = -2^4(2 \cdot 3)(2x+1)^{-4} \qquad f^{iv}(0) = -2^4(2 \cdot 3)$$

$$f^v(x) = 2^5(2 \cdot 3 \cdot 4)(2x+1)^{-5} \qquad f^v(0) = 2^5(2 \cdot 3 \cdot 4)$$

$$\vdots$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n+1} 2^n (n-1)! (2x+1)^{-n} \qquad f^{(n)}(0) = (-1)^{n+1} 2^n (n-1)!$$

$$\text{Hence } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n (n-1)!}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n} x^n$$

b) Use differentiation, as well as your answer in (a) to find the Maclaurin series for

$$\frac{2}{2x+1}.$$

$$\text{Since } \frac{d}{dx} [\ln(2x+1)] = \frac{2}{2x+1}$$

$$\text{Then } \frac{2}{2x+1} = \frac{d}{dx} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n} x^n \right]$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} 2^n x^{n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n 2^{n+1} x^n$$

c) Use the answer from (a) to approximate the definite integral $\int_0^{\frac{1}{5}} \ln(1+2x) dx$ to within five decimal places.

$$\begin{aligned} \int_0^{\frac{1}{10}} \ln(2x+1) dx &= \int_0^{\frac{1}{10}} \left(2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \frac{32}{5}x^5 - \dots \right) dx \\ &= \left[x^2 - \frac{2}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{5}x^5 + \frac{16}{15}x^6 - \dots \right]_0^{\frac{1}{10}} \\ &= \frac{1}{100} - \frac{1}{1500} + \frac{1}{15000} - \frac{1}{125000} + \frac{1}{937500} - \dots \end{aligned}$$

Since $\frac{1}{937500} \approx 0.000001 < 0.000005$, then

$$\int_0^{\frac{1}{10}} \ln(2x+1) dx \approx \frac{1}{100} - \frac{1}{1500} + \frac{1}{15000} - \frac{1}{125000} = \frac{587}{62500} \approx 0.00939$$