MATHEMATICS 201-203-RE
Integral Calculus
Martin Huard
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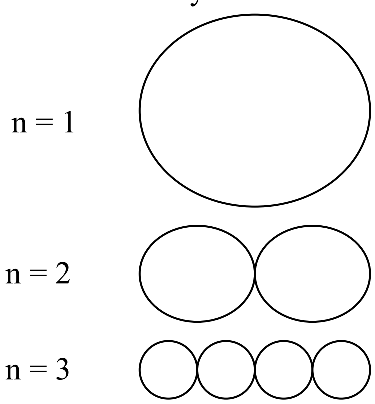
Assignment #4

SOLUTIONS

Question 1 (4 points)

Consider the following sequence of circles. We start with a circle of diameter 2. Next, on the diameter of the first circle, we make two circles. On each of these circles, we again make two circles. We continue this process indefinitely.



a) Let P_n be the perimeter of all the circles after n iteration. Does the sequence P_n converge? If so, to what number?

$$P_1 = 2\pi 2 = 4\pi$$
$$P_2 = 2(2\pi 1) = 4\pi$$
$$P_3 = 4\left(2\pi \tfrac{1}{2}\right) = 4\pi$$
$$\vdots$$
$$P_n = 4\pi$$

Since $\lim_{n\to\infty} P_n = \lim_{n\to\infty} 4\pi = 4\pi$, then P_n converges to 4π

b) Let A_n be the total area occupied by all the circles after n iteration. Does the sequence A_n converge? If so, to what number?

$$A_1 = \pi 2^2 = 4\pi$$
$$A_2 = \pi 1^2 = \pi$$
$$A_3 = \pi\left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}\pi$$
$$\vdots$$
$$A_n = \pi\left(\tfrac{1}{4}\right)^{n-2}$$

Since $\lim_{n\to\infty} A_n = \lim_{n\to\infty} \pi\left(\tfrac{1}{4}\right)^{n-1} = 0$ then A_n converges to 0.

Question 2 (4 points)

Suppose Gauss just won a lottery. He is offered either a lump sum of 10 million dollars now, or \$100 000 a month for life. Assuming money is worth 10% compounded monthly, how many years will it take for the lump sum of 10 million dollars to be equivalent to the \$100 000 a month?

$$P = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$

$$\frac{Pi}{R} = 1-(1+i)^{-n}$$

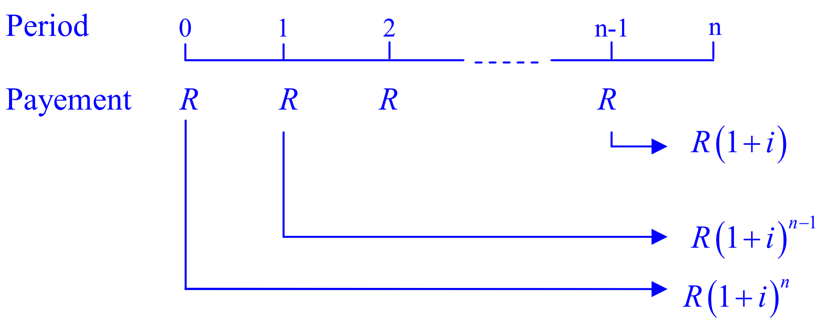
$$(1+i)^{-n} = 1-\frac{Pi}{R}$$

$$-n\ln(1+i) = \ln\left(1-\tfrac{Pi}{R}\right)$$

$$n = \frac{-\ln\left(1-\frac{Pi}{R}\right)}{\ln(1+i)} \qquad n = \frac{-\ln\left(1-\frac{1000000\left(\frac{0.1}{12}\right)}{100000}\right)}{\ln\left(1+\frac{0.1}{12}\right)} \simeq 216 \text{ months} \simeq 18 \text{ years}$$

Question 3 (5 points)

If *\$R* is deposited at the <u>beginning</u> of each period for *n* periods in an account that earns interest at a rate of *i* per period, what will be the balance *S* after *n* periods? This is often referred to as the future value of an **annuity due**.



$$S = R(1+i)^n + R(1+i)^{n-1} + \cdots + R(1+i)^1$$

$$= R(1+i)^n + R(1+i)^{n-1} + \cdots + R(1+i)^1 + R(1+i)^0 - R(1+i)^0$$

$$= \sum_{k=0}^{n} R(1+i)^k - R$$

$$= \frac{R\left(1-(1+i)^{n+1}\right)}{1-(1+i)} - R \qquad \text{geometric series}$$

$$= R\left(\frac{(1+i)^{n+1}-1}{i}\right) - R$$

Question 4 (4 points)

Euler would like to have $400 000 saved by the time he retires. Suppose that he is to deposit money in an account paying 10% interest per year compounded monthly. Also, suppose that he is now 20 years old and plans to retire at 60 years of age.

a) If he starts saving now, how much money should he place in the account each month?

$$S = R\left(\frac{(1+i)^n - 1}{i}\right)$$

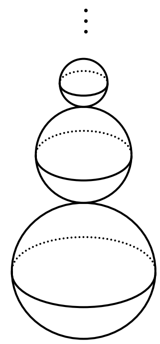
$$R = S\left(\frac{i}{(1+i)^n - 1}\right) = 400\,000\left(\frac{\frac{0.1}{12}}{\left(1+\frac{0.1}{12}\right)^{40\cdot 12} - 1}\right) \simeq \$63.25$$

b) If he will only start saving 20 years from now, how much money will he have to place in the account each month?

$$R = S\left(\frac{i}{(1+i)^n - 1}\right) = 400\,000\left(\frac{\frac{0.1}{12}}{\left(1+\frac{0.1}{12}\right)^{20\cdot 12} - 1}\right) \simeq \$526.75$$

Question 5 (8 points)

Imagine you are stacking an infinite number of spheres of decreasing radii on top of each other, as indicated in the figure. The radii of the spheres are 1 m, $\frac{1}{\sqrt{2}}$ m, $\frac{1}{\sqrt{3}}$ m, etc.



a) How high is this infinite stack of spheres?

$$H = 2\cdot 1 + 2\cdot\tfrac{1}{\sqrt{2}} + 2\cdot\tfrac{1}{\sqrt{3}} + 2\cdot\tfrac{1}{\sqrt{4}} + \cdots$$

$$= \sum_{n=1}^{\infty}\frac{2}{\sqrt{n}} = 2\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$$

This is a *p*-series with $p = \frac{1}{2} < 1$, thus diverges. So the stack is infinitely high.

b) What is the total surface area of all the spheres in the stack?

$$SA = 4\pi\cdot 1^2 + 4\pi\cdot\left(\tfrac{1}{\sqrt{2}}\right)^2 + 4\pi\cdot\left(\tfrac{1}{\sqrt{3}}\right)^2 + 4\pi\cdot\left(\tfrac{1}{\sqrt{4}}\right)^2 + \cdots$$

$$= 4\pi\sum_{n=1}^{\infty}\frac{1}{n}$$

This is the harmonic series, thus diverges. So the surface area is infinite.

c) Show that the volume of the stack is finite.

$$V = \tfrac{4}{3}\pi\cdot 1^3 + \tfrac{4}{3}\pi\cdot\left(\tfrac{1}{\sqrt{2}}\right)^3 + \tfrac{4}{3}\pi\cdot\left(\tfrac{1}{\sqrt{3}}\right)^3 + \tfrac{4}{3}\pi\cdot\left(\tfrac{1}{\sqrt{4}}\right)^3 + \cdots$$

$$= \tfrac{4}{3}\pi\sum_{i=1}^{\infty}\frac{1}{i^{\frac{3}{2}}}$$

This is a *p*-series with $p = \frac{3}{2} > 1$, thus converges. So the volume of the stack is finite.

d) Approximate the volume of the stack with Maple.

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Sum(4*Pi/3*1/n^(3/2),n=1..infinity) =
 evalf(sum(4*Pi/3*1/n^(3/2),n=1..infinity));
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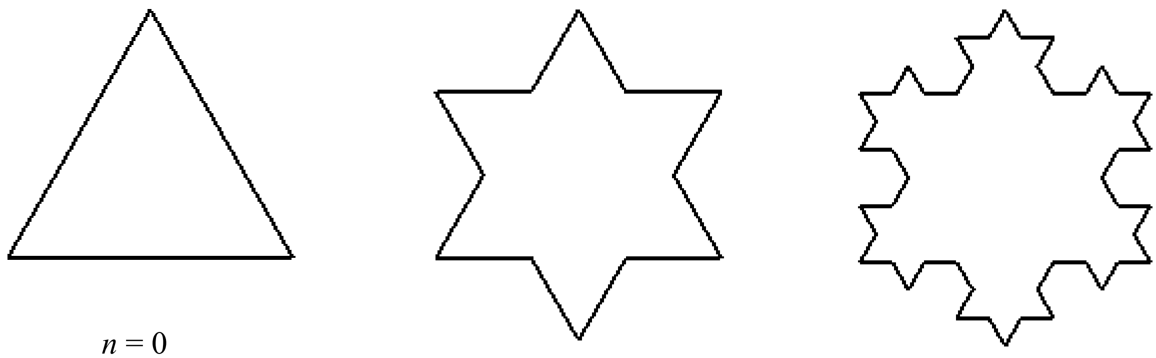
$$\sum_{n=1}^{\infty}\left(\frac{4\,\pi}{3\,n^{(3/2)}}\right) = 10.94269227$$

Thus the stack has a volume of approximately 10.9 m^3.

Question 6 (10 points)

A **sequence of Snowflakes**. Start with an equilateral triangle with sides of length 1. The construction is as follow. The first step is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part. Let us call this snowflake S_1. The second step is to repeat the first step for each side of the resulting polygon to obtain S_2. This process is repeated at each step, obtaining a sequence of snowflakes. The resulting curve is called the **Koch snowflake**, named for Helge von Koch, who described it in 1904.



$n = 0$ $n = 1$ $n = 2$

a) Let s_n, l_n and P_n represent the number of sides, the length of a side and the total length of the nth snowflake. Find formulas for s_n, l_n, and P_n.

n	s_n	l_n	P_n
0	3	1	$3\cdot 1$
1	$4\cdot 3$	$\frac{1}{3}$	$4\cdot 3\cdot\frac{1}{3}$
2	$4^2\cdot 3$	$\frac{1}{9}$	$4^2\cdot 3\cdot\frac{1}{9}$
3	$4^3\cdot 3$	$\frac{1}{27}$	$4^3\cdot 3\cdot\frac{1}{27}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$4^n\cdot 3$	$\frac{1}{3^n}$	$4^n\cdot 3\cdot\frac{1}{3^n}=\frac{4^n}{3^{n-1}}$

Hence

$$s_n = 3\cdot 4^n$$
$$l_n = \frac{1}{3^n}$$
$$P_n = 3\left(4^n\right)\left(\frac{1}{3^n}\right) = 3\left(\frac{4}{3}\right)^n .$$

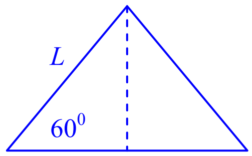
b) Evaluate $\lim_{n\to\infty} P_n$.

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty}\frac{4^n}{3^{n-1}} = \lim_{n\to\infty} 3\left(\frac{4}{3}\right)^n = \infty$$

So the perimeter of the Koch snowflake is infinite

c) Let a_n and t_n represent the area of the smallest triangle and the number of such triangles added on the nth snowflake. Find formulas for a_n and t_n.

Area of an equilateral triangle with sides of length L:



$$A = \tfrac{1}{2} bh = \tfrac{1}{2} L \cdot L \sin 60° = \tfrac{\sqrt{3}}{4} L^2$$

$$a_n = \tfrac{\sqrt{3}}{4} l_n^{\,2} = \tfrac{\sqrt{3}}{4}\left(\tfrac{1}{3^n}\right)^2 = \tfrac{\sqrt{3}}{4}\tfrac{1}{3^{2n}} = \tfrac{\sqrt{3}}{4}\tfrac{1}{9^n}$$

$$t_n = s_{n-1} = 3 \cdot 4^{n-1} \qquad \text{for } n \geq 1$$

d) Let A_n be the area of the nth snowflake. Evaluate $\lim_{n\to\infty} A_n$. [*Hint*, Express A_n as a sum, with a_0 not included in the summation.]

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= a_0 + \sum_{n=1}^{\infty} a_n t_n \\
&= \frac{\sqrt{3}}{4} + \sum_{n=1}^{\infty} \frac{\sqrt{3}}{4}\frac{1}{9^n} 3 \cdot 4^{n-1} \\
&= \frac{\sqrt{3}}{4} + \left(\frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{27} + \cdots\right) \\
&= \frac{\sqrt{3}}{4} + \sum_{n=1}^{\infty} \frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^{n-1} \\
&= \frac{\sqrt{3}}{4} + \frac{\frac{\sqrt{3}}{12}}{1-\frac{4}{9}} \qquad \text{(geometric series with } |r| = \tfrac{4}{9} < 1\text{)} \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} \\
&= \frac{2\sqrt{3}}{5}
\end{aligned}$$

Question 6 (15 points)

Determine if the following series converge.

a) $\sum_{k=1}^{\infty} \frac{k+1}{e^k}$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{k+2}{e^{k+1}}}{\frac{k+1}{e^k}} = \lim_{k\to\infty}\frac{(k+2)e^k}{(k+1)e^{k+1}}\frac{\frac{1}{k}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{1+\frac{2}{k}}{\left(1+\frac{1}{k}\right)e} = \frac{1}{e} < 1$$

Hence $\sum_{k=1}^{\infty} \frac{k+1}{e^k}$ converges by the ratio test.

Note: This could have been done also with the Integral Test

Let $f(x) = \frac{x+1}{e^x}$

1. f is continuous on $[1,\infty)$ (quotient of two continuous function on $\mathbb{R}$, linear and exponential).
2. $f'(x) = \frac{e^x - (x+1)e^x}{(e^x)^2} = \frac{-xe^x}{e^{2x}} = \frac{-x}{e^x} < 0$ for $x \geq 1$, thus f is ↘
3. $f(x) \geq 0$ for $x \geq 1$

$$\begin{aligned}
\int_1^{\infty}\frac{x+1}{e^x}dx &= \lim_{t\to\infty}\int_1^t (x+1)e^{-x}dx \\
&= \lim_{t\to\infty}\left(\left[-(x+1)e^{-x}\right]_1^t + \int_1^t e^{-x}dx\right) \\
&= \lim_{t\to\infty}\left(-(t+1)e^{-t} + 2e^{-1} + \left[-e^{-t}\right]_1^t\right) \\
&= \lim_{t\to\infty}\left(-\tfrac{t+1}{e^t} + 2e^{-1} - e^{-t} + e^{-1}\right) \\
&= \lim_{t\to\infty}\left(-\tfrac{t+1}{e^t}\right) + \lim_{t\to\infty}\left(3e^{-1} - e^{-t}\right) \\
&= \lim_{t\to\infty}\left(-\tfrac{1}{e^t}\right) + \tfrac{3}{e} \qquad \text{H.R. } \tfrac{\infty}{\infty} \\
&= \tfrac{3}{e}
\end{aligned}$$

Since $\int_1^{\infty}\frac{x+1}{e^x}dx$ converges, then $\sum_{k=1}^{\infty}\frac{k+1}{e^k}$ converges by the integral test.

b) $\displaystyle\sum_{n=1}^{\infty}\frac{3n-1}{\sqrt[3]{n^8-n^2+4}}$

Limit Comparison Test

Let $a_n=\dfrac{3n-1}{\sqrt[3]{n^8-n^2+4}}$ and $b_n=\dfrac{n}{\sqrt[3]{n^8}}=\dfrac{n}{n^{\frac{8}{3}}}=\dfrac{1}{n^{\frac{5}{3}}}$

$$\lim_{t\to\infty}\frac{a_n}{b_n}=\lim_{t\to\infty}\frac{\frac{3n-1}{\sqrt[3]{n^8-n^2+4}}}{\frac{1}{n^{\frac{5}{3}}}}=\lim_{t\to\infty}\frac{3n^{\frac{8}{3}}-n^{\frac{5}{3}}}{\sqrt[3]{n^8-n^2+4}}=\lim_{t\to\infty}\frac{n^{\frac{8}{3}}\left(3-\frac{1}{n}\right)}{\sqrt[3]{n^8\left(1-\frac{1}{n^6}+\frac{4}{n^8}\right)}}$$

$$=\lim_{t\to\infty}\frac{n^{\frac{8}{3}}\left(3-\frac{1}{n}\right)}{n^{\frac{8}{3}}\left(1-\frac{1}{n^6}+\frac{4}{n^8}\right)^{\frac{1}{3}}}=\lim_{t\to\infty}\frac{3-\frac{1}{n}}{\left(1-\frac{1}{n^6}+\frac{4}{n^8}\right)^{\frac{1}{3}}}=3$$

Since $\displaystyle\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{5}{3}}}$ converges (p-series with $p>1$) then

$\displaystyle\sum_{n=1}^{\infty}\frac{3n-1}{\sqrt[3]{n^8-n^2+4}}$ converges by the limit comparison test.

c) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n n!}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{3^{n+1}(n+1)!}{(2n+2)!}}{\frac{3^n n!}{(2n)!}}\right|=\lim_{n\to\infty}\frac{3^{n+1}(n+1)!(2n)!}{(2n+2)!3^n n!}$$

$$=\lim_{n\to\infty}\frac{3\left(3^n\right)(n+1)n!(2n)!}{(2n+2)(2n+1)(2n)!3^n n!}$$

$$=\lim_{n\to\infty}\frac{3(n+1)}{(2n+2)(2n+1)}$$

$$=\lim_{n\to\infty}\frac{3}{2(2n+1)}$$

$$=0$$

Thus $\displaystyle\sum_{n=0}^{\infty}\frac{3^n n!}{(2n)!}$ converges by the ratio series test.