**MATHEMATICS 201-203-RE**
Integral Calculus
Martin Huard
Winter 2009

# Assignment #3
## SOLUTIONS

This assignment is due on **Monday March 30, 2009** at the beginning of class.
Complete solutions with exact answers are expected whenever possible.

For questions involving Maple, a print-out of your work is expected, where your name is written in the Worksheet, each question is clearly labeled, and the answers are clearly presented. Also, you must copy your file in my "TEST" subfolder (W:\Tests\mhuard\203\Assignment 3), where your name should be included in the name of the file (for example: Assignment 3 – Your Name).

## Question 1 (13 points)

Consider the integral $\int_1^4 \sqrt{1+x^3}\,dx$. Approximate this integral, with 6 subintervals, using the following methods:

a) Right endpoint approimation

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{1}{2}$$

$$x_i = a + i\Delta x = 1 + \tfrac{1}{2}i$$

$$\begin{aligned}\int_1^4 \sqrt{1+x^3}\,dx \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x \\ &= \left(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)\right)\Delta x \\ &= \tfrac{1}{2}\left(f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)+f(4)\right) \\ &= \tfrac{1}{2}\left(\sqrt{1+1.5^3}+\sqrt{1+2^3}+\sqrt{1+2.5^3}+\sqrt{1+3^3}+\sqrt{1+3.5^3}+\sqrt{1+4^3}\right) \\ &= 14.573304\end{aligned}$$

b) Left endpoint approimation

$$\begin{aligned}\int_1^4 \sqrt{1+x^3}\,dx \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x \\ &= \left(f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\right)\Delta x \\ &= \tfrac{1}{2}\left(f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)\right) \\ &= \tfrac{1}{2}\left(\sqrt{1+1^3}+\sqrt{1+1.5^3}+\sqrt{1+2^3}+\sqrt{1+2.5^3}+\sqrt{1+3^3}+\sqrt{1+3.5^3}\right) \\ &= 11.249282\end{aligned}$$

c) Midpoint Rule

$$\bar{x}_i = \frac{x_{i-1}+x_i}{2} = \frac{1+\frac{1}{2}(i-1)+1+\frac{1}{2}i}{2} = \frac{3}{4}+\frac{1}{2}i$$

$$\int_1^4 \sqrt{1+x^3}\,dx \approx M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\Delta x$$

$$= \left(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+f(\bar{x}_4)+f(\bar{x}_5)+f(\bar{x}_6)\right)\Delta x$$

$$= \tfrac{1}{2}\left(f(1.25)+f(1.75)+f(2.25)+f(2.75)+f(3.25)+f(3.75)\right)$$

$$= \tfrac{1}{2}\left(\sqrt{1+1.25^3}+\sqrt{1+1.5^3}+\sqrt{1+2.25^3}+\sqrt{1+2.75^3}+\sqrt{1+3.25^3}+\sqrt{1+3.75^3}\right)$$

$$= 12.851555$$

d) Trapezoid Rule

$$\int_1^4 \sqrt{1+x^3}\,dx \approx T_6 = \tfrac{\Delta x}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+f(x_6)\right)$$

$$= \tfrac{1}{4}\left(\sqrt{1+1^3}+\sqrt{1+1.5^3}+\sqrt{1+2^3}+\sqrt{1+2.5^3}+\sqrt{1+3^3}+\sqrt{1+3.5^3}+\sqrt{1+4^3}\right)$$

$$= 12.911293$$

e) Simpson Rule

$$\int_1^4 \sqrt{1+x^3}\,dx \approx S_6 = \tfrac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)\right)$$

$$= \tfrac{1}{6}\left(\sqrt{1+1^3}+4\sqrt{1+1.5^3}+2\sqrt{1+2^3}+4\sqrt{1+2.5^3}+2\sqrt{1+3^3}+4\sqrt{1+3.5^3}+\sqrt{1+4^3}\right)$$

$$= 12.871811$$

f) Verify your answers from (a) to (e) using Maple

*with*(*student*);

[D, *Diff*, *Doubleint*, *Int*, *Limit*, *Lineint*, *Product*, *Sum*, *Tripleint*, *changevar*, *completesquare*, *distance*, *equate*, *integrand*, *intercept*, *intparts*, *leftbox*, *leftsum*, *makeproc*, *middlebox*, *middlesum*, *midpoint*, *powsubs*, *rightbox*, *rightsum*, *showtangent*, *simpson*, *slope*, *summand*, *trapezoid*]

*f* := *x* → sqrt(1 + *x*^3);

$$x \to \sqrt{1+x^3}$$

*rightsum*(*f*(*x*), *x* = 1 ..4, 6); *evalf*(%);

$$\frac{1}{2}\sum_{i=1}^{6}\sqrt{1+\left(1+\frac{1}{2}i\right)^3}$$

14.5733038

*leftsum*(*f*(*x*), *x* = 1 ..4, 6); *evalf*(%);

$$\frac{1}{2}\sum_{i=0}^{5}\sqrt{1+\left(1+\frac{1}{2}i\right)^3}$$

$$11.2492817$$

*middlesum*$(f(x), x = 1..4, 6)$; *evalf*$(\%)$;

$$\frac{1}{2}\sum_{i=0}^{5}\sqrt{1+\left(\frac{5}{4}+\frac{1}{2}i\right)^3}$$

$$12.8515552$$

*trapezoid*$(f(x), x = 1..4, 6)$; *evalf*$(\%)$;

$$\frac{1}{4}\sqrt{2}+\frac{1}{2}\sum_{i=1}^{5}\sqrt{1+\left(1+\frac{1}{2}i\right)^3}+\frac{1}{4}\sqrt{65}$$

$$12.9112928$$

*simpson*$(f(x), x = 1..4, 6)$; *evalf*$(\%)$;

$$\frac{1}{6}\sqrt{2}+\frac{1}{6}\sqrt{65}+\frac{2}{3}\sum_{i=1}^{3}\sqrt{1+\left(i+\frac{1}{2}\right)^3}+\frac{1}{3}\sum_{i=1}^{2}\sqrt{1+(1+i)^3}$$

$$12.8718109$$

g) Find the exact value using Maple.

*Int*$(f(x), x = 1..4)$ = *evalf*$($*int*$(f(x), x = 1..4))$;

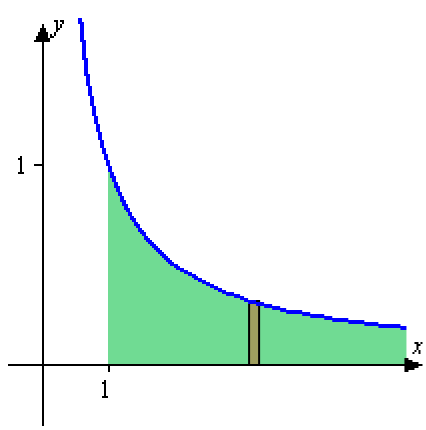
$$\int_1^4 \sqrt{1+x^3}\,dx = 12.8714484$$

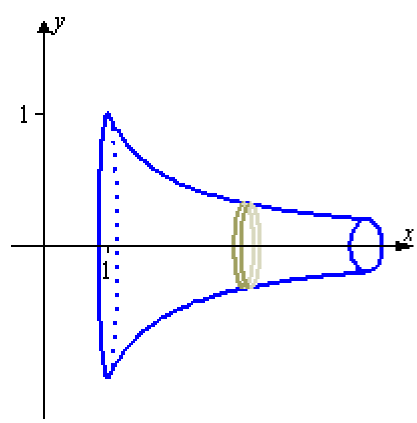
# Question 2 (5 points)

The solid formed by revolving the graph of

$$f(x) = \tfrac{1}{x} \qquad 1 \le x < \infty$$

about the $x$-axis is called **Gabriel's horn**. Find the volume of Gabriel's horn. Sketch the solid, and a typical disk, washer or shell.



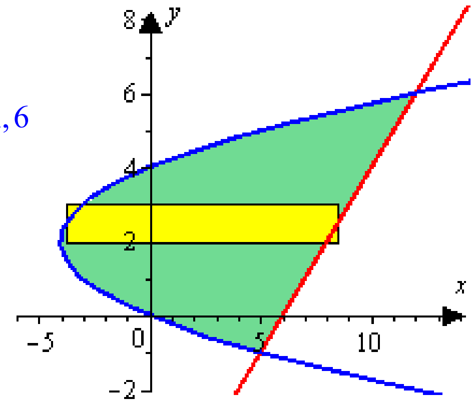

$$V_d = \pi\int_1^\infty \left(\tfrac{1}{x}\right)^2 dx$$
$$= \pi \lim_{t\to\infty}\int_1^t \tfrac{1}{x^2}\,dx$$
$$= \pi \lim_{t\to\infty}\left[\tfrac{-1}{x}\right]_1^t$$
$$= \pi \lim_{t\to\infty}\left(\tfrac{-1}{t}+1\right)$$
$$= \pi$$

# Question 3 (12 points)

Consider the region bounded by $x = y^2 - 4y$ and $y = x - 6$.

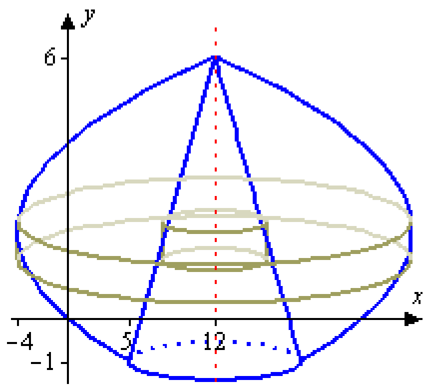
a) Sketch the region and find the area.

Points of intersection:
$$y^2 - 4y = y - 6$$
$$y^2 - 5y + 6 = 0$$
$$(y-6)(y+1) = 0$$
$$y = -1, 6$$



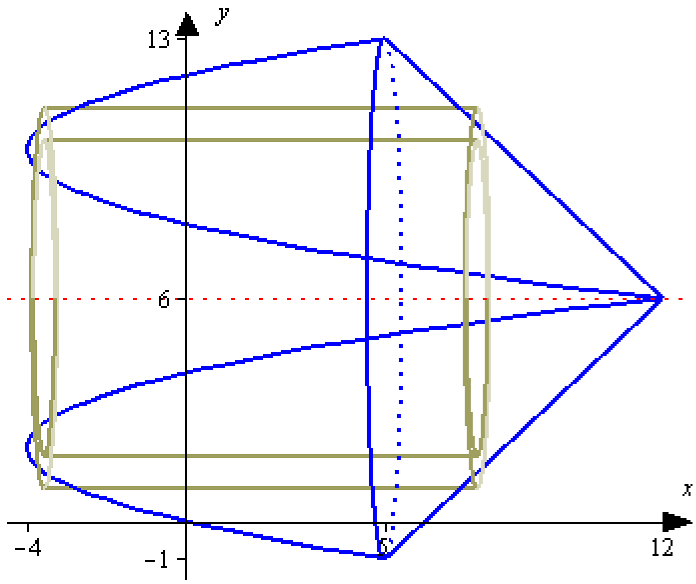

$$A = \int_{-1}^{6} \left(y + 6 - \left(y^2 - 4y\right)\right) dy$$
$$= \int_{-1}^{6} \left(-y^2 + 5y + 6\right) dy$$
$$= \left[\tfrac{-1}{3} y^3 + \tfrac{5}{2} y^2 + 6y\right]_{-1}^{6}$$
$$= (-72 + 90 + 36) - \left(\tfrac{1}{3} + \tfrac{5}{2} - 6\right)$$
$$= \tfrac{343}{6}$$

b) Consider the solid obtained by revolving the given region about the line $x = 12$. Set up the integral to find the volume of the solid obtained by revolving the given region about the line, and use Maple to evaluate it. Sketch the solid, and a typical disk, washer or shell.




$$V_W = \int_{-1}^{6} \pi \left(\left(12 - \left(y^2 - 4y\right)\right)^2 - \left(12 - (y+6)\right)^2\right) dy$$
$$= \frac{4802}{5}\pi$$

c) Consider the solid obtained by revolving the given region about the line $y = 6$.
Set up the integral to find the volume of the solid obtained by revolving the given region about the line, and <u>use Maple</u> to evaluate it. Sketch the solid, and a typical disk, washer or shell.

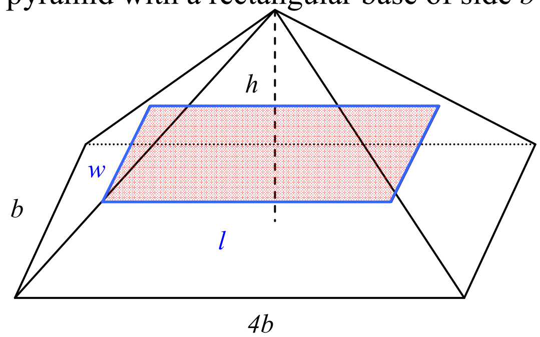



$$V_S = \int_{-1}^{6} 2\pi(6-y)\left(y+6-\left(y^2-4y\right)\right)dy$$

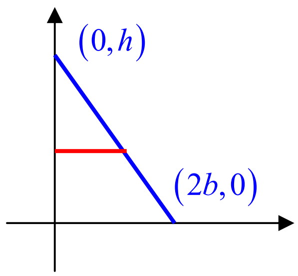
$$= \frac{2401}{6}\pi$$

## Question 4 (6 points)

Find the volume of a pyramid with a rectangular base of side $b$ and $4b$ and of height $h$.




Using rectangular cross sections,




$$m = -\tfrac{h}{2b}$$
$$y = -\tfrac{h}{2b}x + h$$
$$x = -\tfrac{2b}{h}y + 2b$$

$$\begin{aligned} A(y) &= lw \\ &= l\tfrac{1}{4}l \\ &= 2x\tfrac{1}{4}2x \\ &= x^2 \\ &= \left(\tfrac{-2b}{h}y + 2b\right)^2 \end{aligned}$$

$$\begin{aligned} V &= \int_0^h \left(\tfrac{-2b}{h}y + 2b\right)^2 dy \\ &= \left[\tfrac{1}{3}\left(\tfrac{-2b}{h}y + 2b\right)^3\left(\tfrac{-h}{2b}\right)\right]_0^h \\ &= \tfrac{8}{3}b^3\,\tfrac{h}{2b} \\ &= \tfrac{4}{3}hb^2 \end{aligned}$$

# Question 5 (8 points)

Find the consumer surplus and the producer surplus at the equilibrium point for a product with the given demand and supply function.

$$\textit{Demand}:\ p=\sqrt{64-4x} \qquad \textit{Supply}:\ p=x-1$$

Equilibrium quantity: $\sqrt{64-4x}=x-1$

$$64-4x=(x-1)^2$$
$$0=x^2+2x-63$$
$$0=(x-7)(x+9)$$
$$x=-9,7$$

Since $x$ must be non-negative, then $\bar{x}=7$

$$\bar{p}=\sqrt{64-(4)(7)}=6$$

Hence the equilibrium point is (7,6)

Consumer surplus $=\int_0^7\left(\sqrt{64-4x}-6\right)dx$

$$=\left[\tfrac{-1}{4}\tfrac{2}{3}(64-4x)^{\frac{3}{2}}-6x\right]_0^7$$
$$=-\tfrac{1}{6}36^{\frac{3}{2}}-42-\left(-\tfrac{1}{6}64^{\frac{3}{2}}-0\right)$$
$$=\tfrac{22}{3}$$
$$=\$7.33$$

Producer surplus $=\int_0^7\left[6-(x-1)\right]dx$

$$=\int_0^7(7-x)\,dx$$
$$=\left[7x-\tfrac{1}{2}x^2\right]_0^7$$
$$=49-\tfrac{49}{2}=\tfrac{49}{2}$$
$$=\$24.5$$

# Question 6 (6 points)

You are considering buying a franchise that yields a continuous income stream with an annual rate of flow at time $t$ given by $120+10t$ (in thousands of dollars) per year. Find the present value of the franchise (a) for 15 years and (b) forever. Assume that money earns 6% interest per year, compounded continuously.

a) for 15 years

$$P=\int_0^{15}(120+10t)e^{-0.06t}\,dt \qquad u=120+10t \quad v=-\tfrac{50}{3}e^{-0.06t} \qquad du=10dt \quad dv=e^{-0.06t}dt$$

$$=\left[-\tfrac{50}{3}(120+10t)e^{-0.06t}\right]_0^{15}+\tfrac{500}{3}\int_0^{15}e^{-0.06t}\,dt$$

$$=-4500e^{-0.9}+2000-\tfrac{25000}{9}\left[e^{-0.06t}\right]_0^{15}$$

$$=-4500e^{-0.9}+2000-\tfrac{25000}{9}e^{-0.9}+\tfrac{25000}{9}$$

$$\simeq 1818.854$$

Thus the present value is $1 818 854.

b) Forever (capital value)

$$P=\int_0^{\infty}(120+10t)e^{-0.06t}\,dt \qquad u=120+10t \quad v=-\tfrac{50}{3}e^{-0.06t} \qquad du=10dt \quad dv=e^{-0.06t}dt$$

$$=\lim_{a\to\infty}\int_0^{a}(120+10t)e^{-0.06t}\,dt$$

$$=\lim_{a\to\infty}\left(\left[-\tfrac{50}{3}(120+10t)e^{-0.06t}\right]_0^{a}+\tfrac{500}{3}\int_0^{a}e^{-0.06t}\,dt\right)$$

$$=\lim_{a\to\infty}\left(-\tfrac{50}{3}(120+10a)e^{-0.06a}+2000-\tfrac{25000}{9}\left[e^{-0.06t}\right]_0^{a}\right)$$

$$=\lim_{a\to\infty}\left(-2000e^{-0.06a}-\tfrac{500}{3}ae^{-0.06a}-\tfrac{25000}{9}e^{-0.06a}+\tfrac{25000}{9}\right)$$

$$=2000+\tfrac{25000}{9}-\lim_{a\to\infty}\left(2000-\tfrac{25000}{9}\right)e^{-0.06a}-\lim_{a\to\infty}\tfrac{500}{3}ae^{-0.06a}$$

$$=2000+\tfrac{25000}{9}-\lim_{a\to\infty}\frac{500a}{3e^{0.06a}}$$

$$=2000+\tfrac{25000}{9}-\lim_{a\to\infty}\frac{500}{0.18e^{0.06a}} \qquad \text{Hopita's Rule } \tfrac{\infty}{\infty}$$

$$=2000+\tfrac{25000}{9}$$

$$\simeq 4777.778$$

Thus the capital value is $4 777 778.