MATHEMATICS 201-203-RE
Integral Calculus
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Assignment #1
SOLUTIONS

This assignment is due ***Tuesday February 3, 2009*** at the beginning of class.
Complete solutions with exact answers are expected.

For questions involving Maple, a print-out of your work is expected, where your name is written in the Worksheet, each question is clearly labeled, and the answers are clearly presented. Also, you must copy your file in my "TEST" subfolder (W:\Tests\mhuard\203\Assignment 1), where your name should be included in the name of the file (for example: Assignment 1 – Your Name).

Question 1 (5 points)

The weekly marginal cost for producing calculators can be model with $C'(x)=\frac{5}{\sqrt{x}}+10\left(\frac{1}{4}\right)^x+4$, where $C'(x)$ is measured in dollars/unit and x denotes the number of units produced. It has been determined that the weekly fixed cost are \$1000. Find the cost function.

$$\begin{aligned}C(x)&=\int\left(\frac{5}{\sqrt{x}}+\frac{10}{4^x}+4\right)dx\\&=\int\left(5x^{-\frac{1}{2}}+10\left(\tfrac{1}{4}\right)^x+4\right)dx\\&=10\sqrt{x}+\tfrac{10}{\ln\frac{1}{4}}\left(\tfrac{1}{4}\right)^x+4x+D\\&=10\sqrt{x}-\tfrac{5}{\ln 2}\left(\tfrac{1}{4}\right)^x+4x+D\end{aligned}$$

$$\begin{aligned}C(0)&=1000\\-\tfrac{5}{\ln 2}+D&=1000\\D&=1000+\tfrac{5}{\ln 2}\end{aligned}$$

Hence the cost function is:

$$C(x)=10\sqrt{x}-\tfrac{5}{\ln 2}\left(\tfrac{1}{4}\right)^x+4x+1000+\tfrac{5}{\ln 2}$$

Question 2 (10 points)

Evaluate $\int_{-1}^{2} x(x+1)^2\,dx$ using a Riemann Sum.

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = -1 + \frac{3i}{n}$$

$$\begin{aligned}
\int_{-1}^{2} x(x+1)^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(-1+\tfrac{3i}{n}\right)\left(-1+\tfrac{3i}{n}+1\right)^2\tfrac{3}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(-\frac{27i^2}{n^3}+\frac{81i^3}{n^4}\right) \\
&= \lim_{n\to\infty}\left(\frac{-27}{n^3}\sum_{i=1}^{n} i^2+\frac{81}{n^4}\sum_{i=1}^{n} i^3\right) \\
&= \lim_{n\to\infty}\left(\frac{-27}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{81}{n^4}\frac{n^2(n+1)^2}{4}\right) \\
&= \lim_{n\to\infty}\left(\frac{-9(n+1)(2n+1)}{2n^2}+\frac{81(n^2+2n+1)}{4n^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{-18n^2-27n-9}{2n^2}+\frac{81n^2+162n+81}{4n^2}\right) \\
&= \lim_{n\to\infty}\left(-9-\frac{27}{2n}-\frac{9}{2n^2}+\frac{81}{4}+\frac{81}{2n}+\frac{81}{4n^2}\right) \\
&= -9+\frac{81}{4} \\
&= \frac{45}{4}
\end{aligned}$$

Question 3 (6 points)

Evaluate the following limit by first recognizing the sum as a Riemann sum for a function defined on the interval $[0,2]$.

$$\lim_{n\to\infty}\left(\frac{2e^{\frac{2}{n}}}{n}+\frac{2e^{\frac{4}{n}}}{n}+\frac{2e^{\frac{6}{n}}}{n}+\cdots+\frac{2e^{\frac{2n}{n}}}{n}\right)=\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{2e^{\frac{2i}{n}}}{n}$$

$$\Delta x=\frac{2}{n}$$

$$x_i^*=\frac{2i}{n}$$

$$=\lim_{n\to\infty}\sum_{n=1}^{\infty}e^{\frac{2i}{n}}\frac{2}{n}$$

$$=\int_0^2 e^x\,dx$$

$$=\left[e^x\right]_0^2$$

$$=e^2-e^0$$

$$=e^2-1$$

Question 4 (6 points)

If $g(x)=\int_{\tan x}^{\cos x}\sqrt{1+t^3}\,dt$, find $g'(x)$. Verify your answer with Maple.

$$g'(x)=\frac{d}{dx}\left[\int_{\tan x}^{\cos x}\sqrt{1+t^3}\,dt\right]$$

$$=\frac{d}{dx}\left[\int_{\tan x}^{0}\sqrt{1+t^3}\,dt+\int_0^{\cos x}\sqrt{1+t^3}\,dt\right]$$

$$u=\tan x$$

$$v=\cos x$$

$$=-\frac{d}{du}\left[\int_0^u\sqrt{1+t^3}\,dt\right]\frac{du}{dx}+\frac{d}{dv}\left[\int_0^v\sqrt{1+t^3}\,dt\right]\frac{dv}{dx}$$

$$=-\sqrt{1+u^3}\sec^2 x+\sqrt{1+v^3}\,(-\sin x)$$

$$=-\sec^2 x\sqrt{1+\tan^3 x}-\sin x\sqrt{1+\cos^3 x}$$

Question 5 (5 points)

Suppose zinc is being extracted from a certain mine at a rate given by $P'(t)=75\cdot(0.6325)^t$, where P(t) is measured in tons of zinc and t in years. At this rate how much zinc will be extracted during the fourth and fifth years?

$$\int_3^5 75\left(0.6325^t\right)dt=\left.\frac{75\left(0.6325^t\right)}{\ln 0.6325}\right]_3^5$$

$$=\frac{75\left(0.6325^5\right)}{\ln 0.6325}-\frac{75\left(0.6325^3\right)}{\ln 0.6325}$$

$$\simeq 24.86$$

Hence approximately 24.86 tons of zinc was extracted during the fourth and fifth years.

Question 6 (5 points)

The sales of a small clothing store in the first t years of its operation are approximated by the function

$$S(t) = \frac{10(t+2)(t^2+t)}{\sqrt{t}}$$

where t is measured in thousands of dollars. What were the average yearly sales over the first four years of operation?

$$S_{avg} = \frac{1}{4-0}\int_0^4 \frac{10(t+2)(t^2+t)}{\sqrt{t}}\,dt$$
$$= \tfrac{5}{2}\int_0^4 \left(t^{\frac{5}{2}} + 3t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right)dt$$
$$= \tfrac{5}{2}\left[\tfrac{2}{7}t^{\frac{7}{2}} + \tfrac{6}{5}t^{\frac{5}{2}} + \tfrac{4}{3}t^{\frac{3}{2}}\right]_0^4$$
$$= \tfrac{4496}{21} \approx 214.095$$

Hence the average sales were $214 095/year.

Question 7 (4 points)

Use Maple to evaluate the following.

a) $\int_{-1}^{2}\left(x^2 - 3x + 1\right)dx$

b) $\int_0^{\frac{\pi}{3}} \tan^2 x\,dx$

c) $\sum_{i=1}^{m} \frac{1.2}{\ln(i+1)}$ where m is the number obtained from the last three digits of your student number. (For example, if your student number is 0876543, then $m = 543$)

d) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i^4}$

Question 8 (3 points)

Consider the function $f(x) = \frac{x}{x^2+1}$ on the interval $[1,5]$. Use Maple to find the average value of the function.

Question 9 (6 points)

Consider the function $f(x) = -\frac{1}{2}x^2 + 2x + 4$. Using Maple

a) Approximate the area under the curve on the interval $[1,4]$ with 10 rectangles using the right-hand endpoints of the subintervals. Sketch a graph of the function along with the rectangles.

b) Approximate the area under the curve on the interval $[1,4]$ with 10 rectangles using the left-hand endpoints of the subintervals. Sketch a graph of the function along with the rectangles.

c) Approximate the area under the curve on the interval $[1,4]$ with 10 rectangles using the middle of the subintervals. Sketch a graph of the function along with the rectangles.

d) Find the exact area.