MATHEMATICS 201-105-RE
Linear Algebra
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XVI - Basis and Dimension

1. Explain why S is not a basis for $\mathbb{R}^2$. Give which of the two conditions fails.
   a) $S=\{(3,1),(-1,2),(4,1)\}$
   b) $S=\{(3,-5),(-6,10)\}$

2. Explain why S is not a basis for $\mathbb{R}^3$. Give which of the two conditions fails.
   a) $S=\{(1,3,-2),(2,4,1),(5,11,0)\}$
   b) $S=\{(2,3,-5),(1,-6,10)\}$
   c) $S=\{(1,2,3),(5,1,-2),(3,-4,5),(1,-1,1)\}$

3. Explain why S is not a basis for P_2. Give which of the two conditions fails.
   a) $S=\{x^2-1,2x+1,3x^2+6x\}$
   b) $S=\{1+x,5\}$
   c) $S=\{2x^2-3x+1,x^2+x-1,3x^2+2x,x^2+6x-4\}$

4. Explain why S is not a basis for $M_{2,2}$. Give which of the two conditions fails.
   a) $S=\left\{\begin{bmatrix}1&0\\0&2\end{bmatrix},\begin{bmatrix}0&1\\2&0\end{bmatrix}\right\}$
   b) $S=\left\{\begin{bmatrix}1&2\\0&0\end{bmatrix},\begin{bmatrix}0&0\\2&1\end{bmatrix},\begin{bmatrix}1&2\\2&1\end{bmatrix},\begin{bmatrix}1&2\\3&4\end{bmatrix}\right\}$

5. Determine whether the set S is a basis for the indicated vector space. Verify both conditions!
   a) $S=\{(2,5),(-2,5)\}$ for $\mathbb{R}^2$
   b) $S=\{(2,1,0),(1,0,2),(3,1,2)\}$ for $\mathbb{R}^3$
   c) $S=\{(1,2,3,4),(0,1,2,3),(0,0,1,2),(0,0,0,1)\}$ for $\mathbb{R}^4$
   d) $S=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}$ for $M_{2,2}$
   e) $S=\{1+x+x^2,1+3x,x^2-x-5\}$ for P_2

6. Find a basis for $D_{3,3}$ (the vector space of all 3×3 diagonal matrices). What is the dimension of this vector space?

7. Find all subsets of the following set that form a basis for $\mathbb{R}^2$.
   $S=\{(1,0),(0,1),(1,1),(2,2)\}$

8. Find the coordinate vector of $\vec{w}$ relative to the basis S of V.
   a) $\vec{w} = (3,19,2)$ $S = \{(8,11,0),(7,0,10),(1,4,6)\}$ $V = \mathbb{R}^3$
   b) $\vec{w} = (5,-12,3)$ $S = \{(1,2,3),(-4,5,6),(7,-8,9)\}$ $V = \mathbb{R}^3$
   c) $\vec{w} = (11,18,-7)$ $S = \{(4,3,3),(-11,0,11),(0,9,2)\}$ $V = Span(S)$
   d) $\vec{w} = (-1,-5,-2)$ $S = \{(3,-1,4),(5,1,7)\}$ $V = Span(S)$
   e) $\vec{w} = (6,-12,18)$ $S = \{(-1,2,-3)\}$ $V = Span(S)$

9. Find the coordinate vector of $p(x)$ relative to the basis S of V.
   a) $p(x) = 2 - x + x^2$ $S = \{1+x, 1+x^2, x+x^2\}$ $V = P_2$
   b) $p(x) = 3x^2 - 8x - 21$ $S = \{3x^2 - 5, 2x + 4\}$ $V = Span(S)$

10. Find the coordinate vector of $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ relative to the basis

$$S = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

11. Do the following sets W form a basis for the vector space V? Use the dimension of V to facilitate your work.
   a) $W = \{(2,-3,4,1),(1,2,3,-8),(2,1,-5,4)\}$ $V = \mathbb{R}^4$
   b) $W = \{x^2 + x + 1, x + 1\}$ $V = P_2$
   c) $W = \{(2,-3,4),(1,2,3),(2,1,-5)\}$ $V = \mathbb{R}^3$
   d) $W = \{(2,-3,4),(1,2,3),(3,-8,5)\}$ $V = \mathbb{R}^3$
   e) $W = \{(1,-3,4),(1,-5,5),(1,1,1),(2,-4,5)\}$ $V = \mathbb{R}^3$
   f) $W = \left\{ \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \right\}$ $V = M_{2,2}$

12. Extend W to form a basis for $\mathbb{R}^3$ by adding a vector $\vec{w}$ to the set.
   a) $W = \{(1,2,-1),(3,1,0)\}$ b) $W = \{(1,1,1),(1,0,1)\}$

13. Do the following sets W form a basis for $\mathbb{R}^3$? If not, add or subtract vectors from the set to form a basis.
   a) $W = \{(1,2,3)\}$ b) $W = \{(1,2,3),(4,5,6)\}$
   c) $W = \{(1,2,3),(4,5,6),(7,8,9)\}$ d) $W = \{(1,-2,0),(0,3,4),(1,1,1)\}$
   e) $W = \{(1,-2,3),(2,-2,3),(2,-4,6),(1,-4,6)\}$

14. Find a basis for $\operatorname{span}(A)$ and give a geometrical interpretation.

a) $A=\{(1,-2,3),(2,-1,3),(3,-3,6)\}$ b) $A=\{(1,-2,3),(4,5,-6),(7,8,9)\}$

c) $A=\{(1,-2,3),(2,-4,6),(-3,6,-9)\}$

15. For each of the following subspace *W* of $\mathbb{R}^2$,

i) Find a basis for *W*.

ii) Determine the dimension of *W*.

iii) Give a geometrical description of *W*.

a) $W=\{(t,3t):t\in\mathbb{R}\}$ b) $W=\{(a+b+c,2a-4b+c):a,b,c\in\mathbb{R}\}$

16. For each of the following subspace *W* of $\mathbb{R}^3$

i) Find a basis for *W*.

ii) Determine the dimension of *W*.

iii) Give a geometrical description of *W*.

a) $W=\{(t,-t,3t):t\in\mathbb{R}\}$

b) b) $W=\{(t+s,t,s):t,s\in\mathbb{R}\}$

c) $W=\{(0,a+b,2b):a,b\in\mathbb{R}\}$

d) $W=\{(a+b-c,2a+2b-c,-a-b-c):a,b,c\in\mathbb{R}\}$

e) $W=\{(2a-b+5c,a+4b-2c,3a+b+5c)\}$

17. Consider the subspace *W* of $M_{2,2}$ given by $W=\left\{\begin{bmatrix} t & t+s \\ t+s & s\end{bmatrix}:t,s\in\mathbb{R}\right\}$.

a) Find a basis for *W*.

b) Determine the dimension of *W*.

18. Consider the subspace *W* of $M_{2,2}$ given by $W=\left\{\begin{bmatrix} a+b-c & a-b-c \\ 2a+b-2c & 2a-b-2c\end{bmatrix}:a,b,c\in\mathbb{R}\right\}$

a) Find a basis for W.

b) Determine the dimension of *W*.

19. Find a basis and the dimension for the solution space of $AX=0$.

a)
$$\begin{aligned} x+y+z&=0\\ -2x-y+2z&=0\\ -x+3z&=0\end{aligned}$$

b)
$$\begin{aligned} 3x+y+z+w&=0\\ 5x-y+z-w&=0\end{aligned}$$

c)
$$\begin{aligned} x-2y+3z&=0\\ 2x-4y+6z&=0\\ -x+2y-3z&=0\end{aligned}$$

d)
$$\begin{aligned} x-2y+z&=0\\ 3x-6y+z&=0\\ -x+2y+3z&=0\\ 5x-10y-z&=0\end{aligned}$$

e) $3x+2y-5z=0$

Answers

1. a) S is lin. dep. b) S does not span $\mathbb{R}^2$ and S is lin. dep.
2. a) S does not span $\mathbb{R}^3$ and S is lin. dep. b) S does not span $\mathbb{R}^3$ c) S is lin. dep.
3. a) S does not span P_2 and S is lin. dep. b) S does not span P_2
   c) S does not span P_2 and S is lin. dep.
4. a) S does not span $M_{2,2}$ b) S does not span $M_{2,2}$ and S is linearly dependent
5. a) Yes b) No c) Yes d) No e) Yes
6. $S=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$ $\dim\left(D_{3,3}\right)=3$
7. $W_1=\{(1,0),(0,1)\}$, $W_2=\{(1,0),(1,1)\}$, $W_3=\{(1,1),(0,1)\}$, $W_4=\{(1,0),(2,2)\}$ and
   $W_5=\{(0,1),(2,2)\}$
8. a) $(\vec{w})_S=(1,-1,2)$ b) $(\vec{w})_S=(-2,0,1)$ c) $(\vec{w})_S=(0,-1,2)$
   d) $(\vec{w})_S=(3,-2)$ e) $(\vec{w})_S=(-6)$
9. a) $(p(x))_S=(0,2,-1)$ b) $(p(x))_S=(1,-4)$
10. $(A)_S=(-1,1,-1,3)$
11. a) No b) No c) Yes d) No e) No f) No
12. a) $\vec{w}=(1,0,0)$ b) $\vec{w}=(1,0,0)$
13. a) No. $W'=\{(1,2,3),(1,0,0),(0,1,0)\}$ b) No $W'=\{(1,2,3),(4,5,6),(1,0,0)\}$
    c) No. $W'=\{(1,2,3),(4,5,6),(1,0,0)\}$ d) Yes e) No. $W'=\{(1,-2,3),(2,-2,3),(0,0,1)\}$
14. a) $B_{span(A)}=\{(1,-2,3),(2,-1,3)\}$ $\operatorname{span}(A)$ is the plane $x-y-z=0$
    b) $B_{span(A)}=\{(1,-2,3),(4,5,-6),(7,8,9)\}$ $\operatorname{span}(A)=\mathbb{R}^3$
    c) $B_{span(A)}\{(1,-2,3)\}$ $\operatorname{span}(A)$ is the line $x=\dfrac{y}{-2}=\dfrac{z}{3}$
15. a) The line $3x-y=0$ $S=\{(1,3)\}$ $\dim(W)=1$
    b) All $\mathbb{R}^2$ $S=\{(1,2),(1,-4)\}$ $\dim(W)=2$
16. a) The line $x=-y=\frac{z}{3}$ $S=\{(1,-1,3)\}$ $\dim(W)=1$
    b) The plane $x-y-z=0$ $S=\{(1,1,0),(1,0,1)\}$ $\dim(W)=2$
    c) The plane $x=0$ $S=\{(0,1,0),(0,1,2)\}$ $\dim(W)=2$
    d) The plane $3x-2y-z=0$ $S=\{(1,2,-1),(-1,-1,-1)\}$ $\dim(W)=2$
    e) The plane $11x+5y-9z=0$ $S=\{(2,1,3),(-1,4,1)\}$ $\dim(W)=2$
17. $S=\left\{\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix}\right\}$ $\dim(W)=2$
18. $S=\left\{\begin{bmatrix}1&1\\2&2\end{bmatrix},\begin{bmatrix}1&-1\\1&-1\end{bmatrix}\right\}$ $\dim(W)=2$
19. a) $B=\{(3,-4,1)\}$ b) $B=\{(-1,-1,4,0),(0,-1,0,1)\}$
    c) $B=\{(2,1,0),(1,0,1)\}$ d) $B=\{(2,1,0)\}$ e) $B=\{(-2,3,0),(5,0,3)\}$