MATHEMATICS 201-105-RE
Linear Algebra
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XIV - Vector Spaces and Subspaces

1. Describe the zero vector (the additive identity) for the following vector spaces.
   a) $\mathbb{R}^4$ b) $C(-\infty,\infty)$ c) $M_{2,3}$ d) P_3
   e) $V=\{(x,y): x,y\in\mathbb{R},\ x>0\}$ with the following operations :
   $$(x_1,y_1)\oplus(x_2,y_2)=(x_1x_2,y_1+y_2)$$
   $$k\odot(x_1,y_1)=(x_1^{\,k},ky_1)$$

2. Describe the additive inverse of a vector for the following vector spaces.
   a) $\mathbb{R}^4$ b) $C(-\infty,\infty)$ c) $M_{2,3}$ d) P_3
   e) $V=\{(x,y): x,y\in\mathbb{R},\ x>0\}$ with the following operations :
   $$(x_1,y_1)\oplus(x_2,y_2)=(x_1x_2,y_1+y_2)$$
   $$k\odot(x_1,y_1)=(x_1^{\,k},ky_1)$$

3. Determine whether the given set, together with the indicated operations, is a vector space. If it is, prove that each axiom is satisfied, if it is not, identify the axioms that fail.
   a) $M_{2,3}$ with standard operation
   b) $\mathbb{R}^3$ with standard operation
   c) P_3 with the standard operation
   d) The set $\{(x,y): x\geq 0, y\geq 0\}$ with standard operations
   e) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1\\ 1 & b\end{bmatrix}$ with standard operations
   f) The set $\{ax^5 : a\in\mathbb{R}\}$.
   g) $\mathbb{R}^2$ with the following operations : $(x_1,y_1)\oplus(x_2,y_2)=(x_1+x_2,y_1+y_2)$
   $k\odot(x_1,y_1)=(kx_1,y_1)$
   h) $\mathbb{R}^2$ with the following operations : $(x_1,y_1)\oplus(x_2,y_2)=(x_1,0)$
   $k\odot(x_1,y_1)=(kx_1,ky_1)$
   i) $\mathbb{R}^2$ with the following operations : $(x_1,y_1)\oplus(x_2,y_2)=(x_1x_2,y_1y_2)$
   $k\odot(x_1,y_1)=(kx_1,ky_1)$
   j) $\mathbb{R}^2$ with the following operations : $(x_1,y_1)\oplus(x_2,y_2)=(x_1+x_2,y_1+y_2)$
   $k\odot(x_1,y_1)=(k^2x_1,k^2y_1)$

4. Consider the set V whose only element is moon, that is, $V = \{\text{moon}\}$. Is this set a vector space under the following operations?

   moon + moon = moon

   k (moon) = moon for every real number k

5. Determine whether the subset W of $\mathbb{R}^3$, with the standard operations, is a subspace of $\mathbb{R}^3$.
   a) $W = \{(a,b,0) : a,b \in \mathbb{R}\}$
   b) $W = \{(a,1,1) : a \in \mathbb{R}\}$
   c) $W = \{(a,b,a+b) : a,b \in \mathbb{R}\}$
   d) $W = \{(a,b,ab) : a,b \in \mathbb{R}\}$
   e) $W = \{(a,b-a,b) : a,b \in \mathbb{R}\}$
   f) $W = \{(x,y,z) : x-2y+z=0\}$
   g) $W = \{(x,y,z) : 2x+y-z-3=0\}$
   h) $W = \{(x,y,z) : x=2t, y=-t, z=5t,\ t \in \mathbb{R}\}$

6. Determine whether the subset W of $M_{2,2}$ with the standard operations is a subspace of $M_{2,2}$.
   a) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z} \right\}$
   b) W is the set of 2×2 matrices A such that $\det(A) = 0$
   c) W is the set of 2×2 symmetric matrices A
   d) W is the set of diagonal 2×2 matrices.
   e) $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$
   f) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+b+c+d = 0 \right\}$

ANSWERS

1. a) $(0,0,0,0)$ b) $f(x)=0$ c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ d) $p(x)=0+0x+0x^2+0x^3$

   e) (1,0)

2. a) $\vec{u}=(u_1,u_2,u_3,u_4)$ b) $(-p)(x)=-p(x)$ c) $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

   $-\vec{u}=(-u_1,-u_2,-u_3,-u_4)$ $-A=\begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$

   d) $p(x)=a_0+a_1x+a_2x^2+a_3x^3$ e) $\vec{u}=(x_1,y_1)$

   $-p(x)=-a_0-a_1x-a_2x^2-a_3x^3$ $-\vec{u}=\left(\frac{1}{x_1},-y_1\right)$

3. a) Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ and $C=\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$

   1. $A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$ is a 2×3 matrix

   2. $A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}=\begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \end{bmatrix}=B+A$

   3. $$A+(B+C)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}+\begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & b_{13}+c_{13} \\ b_{21}+c_{21} & b_{22}+c_{22} & b_{23}+c_{23} \end{bmatrix}$$
   $$=\begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) & a_{13}+(b_{13}+c_{13}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) & a_{23}+(b_{23}+c_{23}) \end{bmatrix}$$
   $$=\begin{bmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} & (a_{13}+b_{13})+c_{13} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} & (a_{23}+b_{23})+c_{23} \end{bmatrix}$$
   $$=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}+\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$
   $$=(A+B)+C$$

   4. $$A+0_{2\times 3}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
   $$=\begin{bmatrix} a_{11}+0 & a_{12}+0 & a_{13}+0 \\ a_{21}+0 & a_{22}+0 & a_{23}+0 \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}=A$$

5. $$A+\left(-A\right)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}+\begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$$
$$=\begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{12} & a_{13}-a_{13} \\ a_{21}-a_{21} & a_{22}-a_{22} & a_{23}-a_{23} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$=0_{2\times 3}$$

6. $kA=\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix}$ is a 2×3 matrix

7. $$k\left(A+B\right)=\begin{bmatrix} k\left(a_{11}+b_{11}\right) & k\left(a_{12}+b_{12}\right) & k\left(a_{13}+b_{13}\right) \\ k\left(a_{21}+b_{21}\right) & k\left(a_{22}+b_{22}\right) & k\left(a_{23}+b_{23}\right) \end{bmatrix}$$
$$=\begin{bmatrix} ka_{11}+kb_{11} & ka_{12}+kb_{12} & ka_{13}+kb_{13} \\ ka_{21}+kb_{21} & ka_{22}+ka_{22} & ka_{23}+ka_{23} \end{bmatrix}$$
$$=kA+kB$$

8. $$\left(k+l\right)A=\begin{bmatrix} \left(k+l\right)a_{11} & \left(k+l\right)a_{12} & \left(k+l\right)a_{13} \\ \left(k+l\right)a_{21} & \left(k+l\right)a_{22} & \left(k+l\right)a_{23} \end{bmatrix}$$
$$=\begin{bmatrix} ka_{11}+la_{11} & ka_{12}+la_{12} & ka_{13}+la_{13} \\ ka_{21}+la_{21} & ka_{22}+la_{22} & ka_{23}+la_{23} \end{bmatrix}$$
$$=kA+lA$$

9. $$k\left(lA\right)=\begin{bmatrix} k\left(la_{11}\right) & k\left(la_{12}\right) & k\left(la_{13}\right) \\ k\left(la_{21}\right) & k\left(la_{22}\right) & k\left(la_{23}\right) \end{bmatrix}$$
$$=\begin{bmatrix} \left(kl\right)a_{11} & \left(kl\right)a_{12} & \left(kl\right)a_{13} \\ \left(kl\right)a_{21} & \left(kl\right)a_{22} & \left(kl\right)a_{23} \end{bmatrix}$$
$$=\left(kl\right)A$$

10. $$1A=\begin{bmatrix} 1a_{11} & 1a_{12} & 1a_{13} \\ 1a_{21} & 1a_{22} & 1a_{23} \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}=A$$

b) Let $\vec{u}=\left(u_1,u_2,u_3\right)$, $\vec{v}=\left(v_1,v_2,v_3\right)$ and $\vec{w}=\left(w_1,w_2,w_3\right)$

1. $\vec{u}+\vec{v}=\left(u_1+v_1,u_2+v_2,u_3+v_3\right)\in\mathbb{R}^3$
2. $\vec{u}+\vec{v}=\left(u_1+v_1,u_2+v_2,u_3+v_3\right)=\left(v_1+u_1,v_2+u_2,v_3+u_3\right)=\vec{v}+\vec{u}$
3. $$\vec{u}+\left(\vec{v}+\vec{w}\right)=\left(u_1+\left(v_1+v_1\right),u_2+\left(v_2+w_2\right),u_3+\left(v_3+w_3\right)\right)$$
$$=\left(\left(u_1+v_1\right)+w_1,\left(u_2+v_2\right)+w_2,\left(u_3+v_3\right)+w_3\right)$$
$$=\left(\vec{u}+\vec{v}\right)+\vec{w}$$
4. $\vec{u}+\vec{0}=\left(u_1,u_2,u_3\right)+\left(0,0,0\right)=\left(u_1+0,u_2+0,u_3+0\right)=\left(u_1,u_2,u_3\right)=\vec{u}$
5. $\vec{u}+\left(-\vec{u}\right)=\left(u_1,u_2,u_3\right)+\left(-u_1,-u_2,-u_3\right)=\left(u_1-u_1,u_2-u_2,u_3-u_3\right)=\left(0,0,0\right)=\vec{0}$

6. $k\vec{u}=\left(ku_1,ku_2,ku_3\right)\in\mathbb{R}^3$
7. $k\left(\vec{u}+\vec{v}\right)=k\left(u_1+v_1,u_2+v_2,u_3+v_3\right)$
$=\left(k\left(u_1+v_1\right),k\left(u_2+v_2\right),k\left(u_3+v_3\right)\right)$
$=\left(ku_1+kv_1,ku_2+kv_2,ku_3+kv_3\right)$
$=k\vec{u}+k\vec{v}$
8. $\left(k+l\right)\vec{u}=\left(\left(k+l\right)u_1,\left(k+l\right)u_2,\left(k+l\right)u_3\right)$
$=\left(ku_1+lu_1,ku_2+lu_2,ku_3+lu_3\right)$
$=k\vec{u}+l\vec{u}$
9. $k\left(l\vec{u}\right)=k\left(lu_1,lu_2,lu_3\right)=\left(\left(kl\right)u_1,\left(kl\right)u_2,\left(kl\right)u_3\right)=\left(kl\right)\vec{u}$
10. $1\vec{u}=\left(1u_1,1u_2,1u_3\right)=\left(u_1,u_2,u_3\right)=\vec{u}$

c) Let $\mathbf{p}(x)=a_3x^3+a_2x^2+a_1x+a_0$ $\mathbf{q}(x)=b_3x^3+b_2x^2+b_1x+b_0$ and $\mathbf{r}(x)=c_3x^3+c_2x^2+c_1x+c_0$

1. $\mathbf{p}(x)+\mathbf{q}(x)=\left(a_3+b_3\right)x^3+\left(a_2+b_2\right)x^2+\left(a_1+b_1\right)x+\left(a_0+b_0\right)$ is a 3rd degree polynomial
2. $\mathbf{p}(x)+\mathbf{q}(x)=\left(a_3+b_3\right)x^3+\left(a_2+b_2\right)x^2+\left(a_1+b_1\right)x+\left(a_0+b_0\right)$
$=\left(b_3+a_3\right)x^3+\left(b_2+a_2\right)x^2+\left(b_1+a_1\right)x+\left(b_0+a_0\right)$
$=\mathbf{q}(x)+\mathbf{p}(x)$
3. $\mathbf{p}(x)+\left(\mathbf{q}(x)+\mathbf{r}(x)\right)=a_3x^3+a_2x^2+a_1x+a_0+\left(b_3+c_3\right)x^3+\left(b_2+c_2\right)x^2+\left(b_1+c_1\right)x+\left(b_0+c_0\right)$
$=\left(a_3+\left(b_3+c_3\right)\right)x^3+\left(a_2+\left(b_2+c_2\right)\right)x^2+\left(a_1+\left(b_1+c_1\right)\right)x+\left(a_0+\left(b_0+c_0\right)\right)$
$=\left(\left(a_3+b_3\right)+c_3\right)x^3+\left(\left(a_2+b_2\right)+c_2\right)x^2+\left(\left(a_1+b_1\right)+c_1\right)x+\left(\left(a_0+b_0\right)+c_0\right)$
$=\left(a_3+b_3\right)x^3+\left(a_2+b_2\right)x^2+\left(a_1+b_1\right)x+\left(a_0+b_0\right)+c_3x^3+c_2x^2+c_1x+c_0$
$=\left(\mathbf{p}(x)+\mathbf{q}(x)\right)+\mathbf{r}(x)$
4. $\mathbf{p}(x)+\mathbf{0}=\left(a_3+0\right)x^3+\left(a_2+0\right)x^2+\left(a_1+0\right)x+\left(a_0+0\right)$
$=a_3x^3+a_2x^2+a_1x+a_0$
$=\mathbf{p}(x)$
5. $\mathbf{p}(x)+\left(\mathbf{-p}(x)\right)=\left(a_3-a_3\right)x^3+\left(a_2-a_2\right)x^2+\left(a_1-a_1\right)x+\left(a_0+a_0\right)$
$=0x^3+0x^2+0x+0$
$=\mathbf{0}$
6. $k\mathbf{p}(x)=ka_3x^3+ka_2x^2+ka_1x+ka_0$ is a 3rd degree polynomial

7. $k\left(\mathbf{p}(x)+\mathbf{q}(x)\right)=k\left(\left(a_3+b_3\right)x^3+\left(a_2+b_2\right)x^2+\left(a_1+b_1\right)x+\left(a_0+b_0\right)\right)$
$=\left(ka_3+kb_3\right)x^3+\left(ka_2+kb_2\right)x^2+\left(ka_1+kb_1\right)x+\left(ka_0+kb_0\right)$
$=ka_3x^3+ka_2x^2+ka_1x+ka_0+kb_3x^3+kb_2x^2+kb_1x+kb_0$
$=k\mathbf{p}(x)+k\mathbf{q}(x)$
8. $\left(k+l\right)\mathbf{p}(x)=\left(k+l\right)a_3x^3+\left(k+l\right)a_2x^2+\left(k+l\right)a_1x+\left(k+l\right)a_0$
$=\left(ka_3x^3+ka_2x^2+ka_1x+ka_0\right)+\left(la_3x^3+la_2x^2+la_1x+la_0\right)$
$=k\mathbf{p}(x)+l\mathbf{p}(x)$
9. $k\left(l\mathbf{p}(x)\right)=k\left(la_3x^3+la_2x^2+la_1x+la_0\right)=kla_3x^3+kla_2x^2+kla_1x+kla_0=\left(kl\right)\mathbf{p}(x)$
10. $1\mathbf{p}(x)=1a_3x^3+1a_2x^2+1a_1x+1a_0=a_3x^3+a_2x^2+a_1x+a_0=\mathbf{p}(x)$

d) Axiom 5 is not satisfied, there is no $\text{-}\vec{u}$ in the set such that $\vec{u}+\left(\text{-}\vec{u}\right)=\vec{0}$ because
$\text{-}\vec{u}=\left(-1\right)\vec{u}=\left(-u_1,-u_2\right)$ is not in the set.
Axiom 6 is not satisfied because if $k<0$ then $k\vec{u}=\left(ku_1,ku_2\right)$ is not in the set, ku_1 and ku_2 being <0.

e) Axiom 1 is not satisfied since $A+B=\begin{bmatrix}a_{11} & 1\\ 1 & a_{22}\end{bmatrix}+\begin{bmatrix}b_{11} & 1\\ 1 & b_{22}\end{bmatrix}=\begin{bmatrix}a_{11}+b_{11} & 2\\ 2 & a_{22}+b_{22}\end{bmatrix}$ is not in the set.
Axiom 4 is not satisfied since $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is not in the set.
Axiom 5 is not satisfied since $\text{-}A=\begin{bmatrix}-a_{11} & -1\\ -1 & -a_{22}\end{bmatrix}$ is not in the set.
Axiom 6 in not satisfied since $kA=\begin{bmatrix}ka_{11} & k\\ k & ka_{22}\end{bmatrix}$ is not in the set if $k\neq 1$

f) Let $\mathbf{p}(x)=ax^5$, $\mathbf{q}(x)=bx^5$ and $\mathbf{r}(x)=cx^5$.
1. $\mathbf{p}(x)+\mathbf{q}(x)=ax^5+bx^5=\left(a+b\right)x^5$ is in the set
2. $\mathbf{p}(x)+\mathbf{q}(x)=\left(a+b\right)x^5=\left(b+a\right)x^5=\mathbf{q}(x)+\mathbf{p}(x)$
3. $\mathbf{p}(x)+\left(\mathbf{q}(x)+\mathbf{r}(x)\right)=ax^5+\left(b+c\right)x^5=\left(a+\left(b+c\right)\right)x^5=\left(\left(a+b\right)+c\right)x^5$
$=\left(a+b\right)x^5+cx^5=\left(\mathbf{p}(x)+\mathbf{q}(x)\right)+\mathbf{r}(x)$
4. $\mathbf{p}(x)+\mathbf{0}=\left(a+0\right)x^5=ax^5=\mathbf{p}(x)$
5. $\mathbf{p}(x)+\left(\text{-}\mathbf{p}(x)\right)=\left(a-a\right)x^5=0x^5=\mathbf{0}$
6. $k\mathbf{p}(x)=kax^5$ is in the set
7. $k\left(\mathbf{p}(x)+\mathbf{q}(x)\right)=k\left(\left(a+b\right)x^5\right)=\left(ka+kb\right)x^5=kax^5+kbx^5=k\mathbf{p}(x)+k\mathbf{q}(x)$

8. $(k+l)\mathbf{p}(x) = (k+l)ax^5 = kax^5 + lax^5 = k\mathbf{p}(x) + l\mathbf{p}(x)$
9. $k(l\mathbf{p}(x)) = k(lax^5) = klax^5 = (kl)\mathbf{p}(x)$
10. $1\mathbf{p}(x) = 1ax^5 = ax^5 = \mathbf{p}(x)$

g) The set is not a vector space since axiom 8 fails. For example, let $k=1,\ l=2$ and $\vec{u}=(1,1)$.

$$(k+l)\odot\vec{u} = (1+2)\odot(1,1) = ((1+2)1,1) = (3,1)$$

$$(k\odot\vec{u})\oplus(l\odot\vec{u}) = (1\odot(1,1))\oplus(2\odot(1,1)) = (1,1)\oplus(2,1) = (3,2)$$

Thus $(k+l)\odot\vec{u} \neq (k\odot\vec{u})\oplus(l\odot\vec{u})$.

Axioms 4 and 5 also fail.

h) The set is not a vector space because axiom 2 fails. For example, let $\vec{u}=(1,2)$ and $\vec{v}=(2,1)$.

$$\vec{u}\oplus\vec{v} = (1,2)\oplus(2,1) = (1,0)$$

$$\vec{v}\oplus\vec{u} = (2,1)\oplus(1,2) = (2,0)$$

Thus $\vec{u}\oplus\vec{v} \neq \vec{v}\oplus\vec{u}$.

Axioms 4, 5 and 8 also fail.

i) Axiom 4 fails since $\vec{u}\oplus\vec{0} = (u_1\cdot 0, u_2\cdot 0) = (0,0) \neq \vec{u}$ if $\vec{u}\neq\vec{0}$

Axiom 5 and 7 also fail.

j) Axiom 8 fails

$$\begin{aligned}(k+l)\odot\vec{u} &= \left((k+l)^2u_1, (k+l)^2u_2, (k+l)^2u_3\right)\\ &= \left(k^2u_1 + 2klu_1 + l^2u_1, k^2u_2 + 2klu_2 + l^2u_2, k^2u_3 + 2klu_3 + l^2u_3\right)\\ &= (k\odot\vec{u})\oplus\left(\sqrt{2kl}\odot\vec{u}\right)\oplus(l\odot\vec{u})\\ &\neq (k\odot\vec{u})\oplus(l\odot\vec{u})\end{aligned}$$

Axiom 5 also fails.

4. Yes. It is similar to the vector space $V=\{\vec{0}\}$.

5. a) Yes 1. $\vec{u}+\vec{v} = (u_1,u_2,0)+(v_1,v_2,0) = (u_1+v_1, u_2+v_2, 0)\in W$

2. $k\vec{u} = k(u_1,u_2,0) = (ku_1,ku_2,0)\in W$

Thus W is a subspace of $\mathbb{R}^3$

b) No 1. $\vec{u}+\vec{v} = (u_1,1,1)+(v_1,1,1) = (u_1+v_1,2,2)\notin W$

c) Yes 1. $\vec{u}+\vec{v} = (u_1,u_2,u_1+u_2)+(v_1,v_2,v_1+v_2) = (u_1+v_1, u_2+v_2, (u_1+v_1)+(u_2+v_2))\in W$

2. $k\vec{u} = k(u_1,u_2,u_1+u_2) = (ku_1,ku_2,ku_1+ku_2)\in W$

Thus W is a subspace of $\mathbb{R}^3$

d) No 2. $k\vec{u} = k(u_1, u_2, u_1u_2) = (ku_1, ku_2, ku_1u_2) \notin W$ since $(ku_1)(ku_2) = k^2u_1u_2 \neq ku_1u_2$ if $k \neq 1$.

e) Yes 1. $\vec{u} + \vec{v} = (u_1, u_2 - u_1, u_2) + (v_1, v_2 - v_1, v_2) = (u_1 + v_1, (u_2 + v_2) - (u_1 + v_1), u_2 + v_2) \in W$

2. $k\vec{u} = k(u_1, u_2 - u_1, u_2) = (ku_1, ku_2 - ku_1, u_2) \in W$

Thus W is a subspace of $\mathbb{R}^3$

f) Yes If $\vec{u} \in W$ then $u_1 - 2u_2 + u_3 = 0$ and if $\vec{v} \in W$ then $v_1 - 2v_2 + v_3 = 0$.

1. $\vec{u} + \vec{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_v)$

Since $(u_1 + v_1) - 2(u_2 + v_2) + (u_3 + v_3) = (u_1 - 2u_2 + u_3) + (v_1 - 2v_2 + v_3) = 0 + 0 = 0$ then $\vec{u} + \vec{v} \in W$.

2. $k\vec{u} = k(u_1, u_2, u_3) = (ku_1, ku_2, ku_3)$

Since $ku_1 - 2ku_2 + ku_3 = k(u_1 - 2u_2 + u_3) = k0 = 0$ then $k\vec{u} \in W$.

Thus W is a subspace of $\mathbb{R}^3$

g) No If $\vec{u} \in W$ then $2u_1 + u_2 - u_3 = 3$ and if $\vec{v} \in W$ then $2v_1 + v_2 - v_3 = 3$.

1. $\vec{u} + \vec{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_v)$

Since $2(u_1 + v_1) + (u_2 + v_2) - (u_3 + v_3) = (2u_1 + u_2 - u_3) + (2v_1 + v_2 - v_3) = 3 + 3 = 6$ then $\vec{u} + \vec{v} \notin W$.

h) Yes If $\vec{u} \in W$ then $\vec{u} = (2t, -t, 5t)$ and if $\vec{v} \in W$ then $\vec{v} = (2s, -s, 5s)$.

1. $\vec{u} + \vec{v} = (2t, -t, 5t) + (2s, -s, 5s)$

$= (2t + 2s, -t - s, 5t + 5s) = (2(t + s), -(t + s), 5(t + s)) \in W$

2. $k\vec{u} = k(2t, -t, 5t) = (2kt, -kt, 5kt) = (2(kt), -(kt), 5(kt)) \in W$

Thus W is a subspace of $\mathbb{R}^3$

6. a) No. We do not always have closure under scalar multiplication.

For a example if $k = \frac{1}{2}$ and A=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in W$, then $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \notin W$

b) No. Let A=$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det(A) = \det(B) = 0$ so $A, B \in W$.

Since $\det(A + B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$, then $A + B \notin W$, so we do not have closure under addition.

c) Yes. Let A and B be symmetric matrices, $A^T = A$ and $B^T = B$.

1. $(A + B)^T = A^T + B^T = A + B$, hence $A + B \in W$

2. $(kA)^T = kA^T = kA$, hence $kA \in W$

Hence W is a subspace of $M_{2,2}$.

d) Yes. Let $A=\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$ and $B=\begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}$ be in W.

1. $A+B=\begin{bmatrix} a_{11}+b_{11} & 0 \\ 0 & a_{22}+b_{22} \end{bmatrix} \in W$

2. $kA=\begin{bmatrix} ka_{11} & 0 \\ 0 & ka_{22} \end{bmatrix} \in W$

Hence W is a subspace of $M_{2,2}$.

e) Yes 1. $A+B=\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & a_{22}+b_{22} \end{bmatrix} \in W$

2. $kA=k\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}=\begin{bmatrix} ka_{11} & ka_{12} \\ 0 & ka_{22} \end{bmatrix} \in W$

Hence W is a subspace of $M_{2,2}$.

f) Yes 1. $A+B=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \in W$ since

$$a_{11}+b_{11}+a_{12}+b_{12}+a_{21}+b_{21}+a_{22}+b_{22}=\left(a_{11}+a_{12}+a_{21}+a_{22}\right)+\left(b_{11}+b_{12}+b_{21}+b_{22}\right)=0$$

2. $kA=k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} ka_{11} & ka_{12} \\ ka_{22} & ka_{22} \end{bmatrix} \in W$ since

$$ka_{11}+ka_{12}+ka_{21}+ka_{22}=k\left(a_{11}+a_{12}+a_{21}+a_{22}\right)=0$$

Hence W is a subspace of $M_{2,2}$.