MATHEMATICS 201-105-RE
Linear Algebra
Martin Huard
Winter 2006

XIII - Planes in $\mathbb{R}^3$

1. Find, if possible, an equation for the plane in
   i) vector form
   ii) point-normal form
   iii) general form

   a) Passing through P(5,-1,2) and parallel to the vectors $\vec{u} = (2,7,-4)$ and $\vec{v} = (2,-1,5)$.
   b) Passing through the points P(-2,1,5), Q(-2,1,-3) and R(1,1,4).
   c) Passing through the points P(1,0,6), Q(-3,4,7) and R(2,0,12).
   d) Passing through the point P(2,-5,5) and having $\vec{n} = (1,-2,1)$.
   e) Passing through the point P(3,-1,8) and containing the line $L:(x,y,z) = (-1,2,5) + t(1,1,-2) \quad t \in \mathbb{R}$.
   f) Containing the lines $L_1:\begin{cases} x = 6+4t \\ y = 2-t \\ z = 4+3t \end{cases} \quad t \in \mathbb{R}$ and $L_2: \dfrac{x-2}{3} = \dfrac{3-y}{2} = z-1$.
   g) Containing the lines $L_1:(x,y,z) = (-1,2,4) + t(4,-1,3)$, $t \in \mathbb{R}$ and $L_2: \dfrac{x-2}{3} = \dfrac{3-y}{2} = z-1$.
   h) Containing the lines $L_1:(-1,2,2) + t(2,1,-3) \quad t \in \mathbb{R}$ and $L_2: \dfrac{x-2}{2} = y-1 = \dfrac{2-z}{3}$.
   i) Passing through the point P(3,-2,5) and perpendicular to the line $L:(x,y,z) = (-1,2,2) + t(3,2,1) \quad t \in \mathbb{R}$.
   j) Containing the line $L_1: \dfrac{x+3}{2} = y+5 = 2z$ and perpendicular to the plane $\pi: 2x-4y+z-1=0$.
   k) Passing through P(2,-1,4) and parallel to the plane $\pi: 3x-4y+z-5=0$
   l) Passing through P(2,-3,5) and parallel to the *xz*-plane.
   m) Passing through P(-1,1,5) and containing the intersection of the planes $\pi_1: 15x+y+9z=62$ and $\pi_2:\begin{cases} x = 6+5s+3t \\ y = -10+8s+t \\ z = -2+3s+t \end{cases} \quad s,t \in \mathbb{R}$.
   n) Parallel to the *yz*-plane and passing through the point P of intersection between the line $L:(x,y,z) = (-1,3,3) + t(3,1,5)$, $t \in \mathbb{R}$ and the plane $\pi: 2x-4y+z-10=0$.

2. Consider the following planes.

$$\pi_1 : x - 2y - z - 4 = 0$$

$$\pi_2 : (x, y, z) = (0, 2, -1) + r(-1, 3, 3) + t(1, 1, 0) \qquad s, t \in \mathbb{R}$$

$$\pi_3 : \begin{cases} x = -2 + 4s - 2t \\ y = s - 3t \\ z = 5 + 2s + 4t \end{cases} \qquad s, t \in \mathbb{R}$$

$$\pi_4 : 2(x-1) + 3(y-5) - 4(z+1) = 0$$

   a) Find the angle between the planes π_1 and π_2, the planes π_1 and π_3 and the planes π_1 and π_4.
   b) Find which planes are parallel or perpendicular with π_1.
   c) Find, if possible, the intersection of the planes π_1 and π_2, the planes π_1 and π_3 and the planes π_1 and π_4.
   d) Determine the distance from each plane to the point P(1,2,-3).
   e) Determine the distance between the pair of planes π_1 and π_3.
   f) For each of the points A(1,7,2), B(15,9,-7) and C(1,1,1), determine to which plane (if any) they belong.
   g) Find the point on each plane that is closest to the point P(1,4,2).
   h) For each plane, find the equation of the line passing through the point P(1,2,-3) and perpendicular to the plane.

3. Prove that the distance from the origin to the plane $\pi : ax + by + cz + d = 0$ is given by

$$d(0, \pi) = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

4. Prove that the distance from $P_0(x_0, y_0, z_0)$ to the plane $\pi : ax + by + cz + d = 0$ is given by

$$d(P_0, \pi) = \frac{|ax_0 + by_0 + cz_0 d|}{\sqrt{a^2 + b^2 + c^2}}$$

5. Prove that the distance between the parallel planes $\pi_1 : ax + by + cz + d_1 = 0$ and $\pi_2 : ax + by + cz + d_2 = 0$ is given by

$$d(\pi_1, \pi_2) = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

6. Consider the line $L : \dfrac{x-1}{3} = \dfrac{2-y}{2} = z + 1$

   a) Find the equation of the plane perpendicular to L passing through $P(2, -1, 3)$.
   b) Find the intersection between L and the plane found in (a).

7. Consider the plane $\pi : 2x - 3y + z - 3 = 0$.
   a) Find the equation of the line passing through the point $P(3,-1,5)$ and perpendicular to the plane π.
   b) Find the point Q on the plane π that is closest to the point $P(3,-1,5)$.
   c) Find the distance between P and the plane π.
   d) Find the equation for the line passing through the point $P(3,-1,5)$ and parallel to the planes π and $\pi_2 : x - y + 3z - 5 = 0$.
   e) Find the equation of the plane parallel to π and passing through the point $P(3,-1,5)$.

Answers

1. a) (i) $(x,y,z) = (5,-1,2) + s(2,7,-4) + t(2,-1,5) \quad s,t \in \mathbb{R}$
   (ii) $31(x-5) - 18(y+1) - 16(z-2) = 0$ (iii) $31x - 18y - 16z - 141 = 0$
   b) (i) $(x,y,z) = (-2,1,5) + s(0,0,-8) + t(3,0,-1) \quad s,t \in \mathbb{R}$
   (ii) $-24(y-1) = 0$ (iii) $y - 1 = 0$
   c) (i) $(x,y,z) = (1,0,6) + s(-4,4,1) + t(1,0,6) \quad s,t \in \mathbb{R}$
   (ii) $24(x-1) + 25y - 4(z-6) = 0$ (iii) $24x + 25y - 4z = 0$
   d) (ii) $(x-2) - 2(y+5) + (z-5) = 0$ (iii) $x - 2y + z - 17 = 0$
   (i) $(x,y,z) = (2,-5,5) + s(2,1,0) + t(-1,0,1) \quad s,t \in \mathbb{R}$
   e) (i) $(x,y,z) = (3,-1,8) + s(1,1,-2) + t(-4,3,-3) \quad s,t \in \mathbb{R}$
   (ii) $3(x-3) + 11(y+1) + 7(z-8) = 0$ (iii) $3x + 11y + 8z - 54 = 0$
   f) L_1 and L_2 intersect at P(2,3,1) (i) $(x,y,z) = (2,3,1) + s(4,-1,3) t(3,-2,1) \quad s,t \in \mathbb{R}$
   (ii) $5(x+1) + 5(y-2) - 5(z-5)$ (iii) $x + y - z - 4 = 0$
   g) L_1 and L_2 do not intersect, thus there are no planes containing the lines L_1 and L_2.
   h) (i) $(x,y,z) = (-1,2,2) + s(3,-1,0) + t(2,1,-3) \quad s,t \in \mathbb{R}$
   (ii) $3(x+1) + 9(y-2) + 5(z-2) = 0$ (iii) $3x + 9y + 5z - 25 = 0$
   i) (ii) $3(x-3) + 2(y+2) + (z-5) = 0$ iii) $3x + 2y + z - 10 = 0$
   (iii) $(x,y,z) = (3,-2,5) + s(-2,3,0) + t(-1,0,3) \quad s,t \in \mathbb{R}$
   j) (i) $(x,y,z) = (-3,-5,0) + s(2,1,\tfrac{1}{2}) + t(2,-4,1) \quad s,t \in \mathbb{R}$
   (ii) $3(x+3) - (y+5) - 10z = 0$ (iii) $3x - y - 10z + 4 = 0$

k) (ii) $3(x-2)-4(y+1)+(z-4)=0$ (iii) $3x-4y+z-14=0$

(i) $(x,y,z)=(2,-1,4)+s(4,3,0)+t(-1,0,3) \quad s,t\in\mathbb{R}$

l) (i) $(x,y,z)=(2,-1,4)+s(1,0,0)+t(0,0,1) \quad s,t\in\mathbb{R}$

(ii) $0(x-2)-(y+1)+0(z-4)=0$ (iii) $y=-1$

m) $\pi_1\cap\pi_2 : (x,y,z)=(4,2,0)+t(-1,6,1)$, thus taking the points (4,2,0) and (3,8,1) from $\pi_1\cap\pi_2$ we have (i) $(x,y,z)=(-1,1,5)+s(5,1,-5)+t(4,7,-4) \quad s,t\in\mathbb{R}$

(ii) $31(x+1)+31(z-5)=0$ (iii) $x+z-4=0$

l) $\pi\cap L=\{(8,6,18)\}$ (i) $(x,y,z)=(8,6,18)+s(0,1,0)+t(0,0,1) \quad s,t\in\mathbb{R}$

(ii) $1(x-8)+0(y-6)+0(z-18)=0$ (iii) $x=8$

2. a) 69.5° 0° 90°

b) $\pi_1 // \pi_3$ and $\pi_1 \perp \pi_4$.

c) $\pi_1\cap\pi_2 : (x,y,z)=\left(-\frac{32}{3},-\frac{22}{3},0\right)+t\left(-\frac{11}{3},-\frac{7}{3},1\right) \quad t\in\mathbb{R}$ $\pi_1\cap\pi_3:\varnothing$

$\pi_1\cap\pi_4 : (x,y,z)=\left(-\frac{30}{7},-\frac{29}{7},0\right)+t\left(\frac{11}{7},\frac{2}{7},1\right) \quad t\in\mathbb{R}$

d) $\frac{2\sqrt{6}}{3}$ $\frac{5\sqrt{34}}{34}$ $\frac{7\sqrt{6}}{6}$ $\frac{\sqrt{29}}{29}$

e) $\frac{11\sqrt{6}}{6}$

f) $A\in\pi_2$ $B\in\pi_1$ and $B\in\pi_2$ C is not on any of the planes

g) $\left(\frac{19}{6},\frac{-1}{3},\frac{-1}{6}\right)$ $\left(\frac{7}{34},\frac{163}{34},\frac{16}{17}\right)$ $\left(\frac{4}{3},\frac{10}{3},\frac{5}{3}\right)$ $\left(\frac{59}{29},\frac{161}{29},\frac{-2}{29}\right)$

h) $x-1=\frac{2-y}{2}=-z-3$ $\frac{1-x}{3}=\frac{y-2}{3}=\frac{-z-3}{4}$ $x-1=\frac{2-y}{2}-z-3$ $\frac{x-1}{2}=\frac{y-2}{3}=\frac{-z-3}{4}$

3. We have $\vec{n}=(a,b,c)$. Let us take $R\left(-\frac{d}{a},0,0\right)$.

$$d(0,\pi)=\frac{\left|\overrightarrow{RP}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(-\frac{d}{a}\right)\bullet(a,b,c)\right|}{\sqrt{a^2+b^2+c^2}}=\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

4. We have $\vec{n}=(a,b,c)$. Let us take $R\left(-\frac{d}{a},0,0\right)$.

$$d(P_0,\pi)=\frac{\left|\overrightarrow{RP_0}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(x_0+\frac{d}{a},y_0,z_0\right)\bullet(a,b,c)\right|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

5. We have $\vec{n}=(a,b,c)$. Let us take $P_1\left(-\frac{d_1}{a},0,0\right)$ and $P_2\left(-\frac{d_2}{a},0,0\right)$.

$$d(\pi_1,\pi_2)=\frac{\left|\overrightarrow{P_1P_2}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(-\frac{d_2}{a}+\frac{d_1}{a},0,0\right)\bullet(a,b,c)\right|}{\sqrt{a^2+b^2+c^2}}=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

6. a) $3x-2y+z-11=0$ b) $\left(\frac{53}{14},\frac{1}{7},\frac{-1}{14}\right)$

7. a) $\dfrac{x-3}{2}=\dfrac{-y-1}{3}=z-5$ b) $\left(\frac{10}{7},\frac{19}{14},\frac{59}{14}\right)$ c) $\frac{11\sqrt{14}}{14}$

d) $\dfrac{3-x}{8}=\dfrac{-y-1}{5}=z-5$ e) $2x-3y+z-14=0$