

MATHEMATICS 201-105-RE

Linear Algebra

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XI - Lines in \mathbb{R}^2

1. Find, if possible, an equation for the line in

- i) vector form
- ii) parametric form
- iii) symmetric form
- iv) general form
- v) slope-intercept form ($y = mx + b$)

- a) Passing through $P(-1,2)$ and parallel to the vector $\vec{u} = (-2,3)$.
- b) Passing through the points $P(2,5)$ and $Q(1,-3)$.
- c) Passing through the points $P(4,-1)$ and $Q(2,2)$.
- d) Passing through the points $P(2,-3)$ and $Q(4,-3)$.
- e) Passing through $P(3,-4)$ and perpendicular to the vector $\vec{n} = (1,-5)$.
- f) Passing through the point $P(1,1)$ and parallel to $y = 3x - 5$.
- g) Passing through the point $P(2,4)$ and parallel to $\frac{x-1}{2} = \frac{3-y}{5}$.
- h) Passing through the point $P(3,-2)$ and parallel to $2x - 5y + 4 = 0$.
- i) Passing through P and Q , where P is the intersection of the lines $L_1 : \frac{x+3}{2} = y+5$ and $L_2 : (x, y) = (2, 2) + t(1, 2)$, $t \in \mathbb{R}$ and Q is the intersection of the lines $L_3 : 5x - 2y - 3 = 0$ and $L_4 : \begin{cases} x = 7 + 6t \\ y = -4 - 5t \end{cases} \quad t \in \mathbb{R}$.
- j) Passing through $P(2,0)$ and tangent to the circle $C : x^2 + y^2 = 2$.

2. Consider the following lines.

$$L_1 : \frac{x+3}{2} = y+5$$

$$L_2 : (x, y) = (2, 2) + t(-2, 4), \quad t \in \mathbb{R}$$

$$L_3 : 2x + y - 6 = 0$$

$$L_4 : \begin{cases} x = 7 + t \\ y = -4 \end{cases} \quad t \in \mathbb{R}$$

- a) Find the angle between the lines
 - i) L_1 and L_2
 - ii) L_1 and L_3
 - iii) L_1 and L_4
 - iv) L_2 and L_3 .
- b) Find which pair of lines are parallel and which are perpendicular.
- c) Find, if possible, the intersection of the lines
 - i) L_1 and L_2
 - ii) L_1 and L_3
 - iii) L_1 and L_4

- d) Determine the distance from each line to the point $P(2,-3)$.
- e) Determine the distance between each pair of parallel lines found in (b).
- f) For each of the points $A(-2,1)$, $B(1,4)$ and $C(-3,-4)$, determine to which line they belong.
3. Consider the lines $L_1 : ax + y - 3 = 0$ and $L_2 : 9x + ay - 9 = 0$ Find the value of a such that the lines are
- a) The same b) Parallel and distinct
c) Perpendicular c) Neither parallel nor perpendicular.
4. Prove that the distance between the origin and the line given by the equation $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2 + b^2}}$.
5. Prove that the distance between the point $P(x_0, y_0)$ and the line given by the equation $ax + by + c = 0$ is $\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$.
6. Prove that the distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.
7. Find the equation of all lines parallel to the line $L : 3x - 4y + 7 = 0$ with a distance of 3 units from it.

Answers

1. a) $(x, y) = (-1, 2) + t(-2, 3), t \in \mathbb{R}, \begin{cases} x = -1 - 2t \\ y = 2 + 3t \end{cases} t \in \mathbb{R}, \frac{x+1}{-2} = \frac{y-2}{3},$
 $3x + 2y - 1 = 0, y = -\frac{3}{2}x + \frac{1}{2}$
- b) $(x, y) = (2, 5) + t(-1, -8), t \in \mathbb{R}, \begin{cases} x = 2 - t \\ y = 5 - 8t \end{cases} t \in \mathbb{R}, 2 - x = \frac{5 - y}{8},$
 $8x - y - 11 = 0, y = 8x - 11$
- c) $(x, y) = (4, -1) + t(-2, 3), t \in \mathbb{R}, \begin{cases} x = 4 - 2t \\ y = -1 + 3t \end{cases} t \in \mathbb{R}, \frac{x-4}{-2} = \frac{y+1}{3},$
 $3x + 2y - 10 = 0, y = -\frac{3}{2}x + 5$
- d) $(x, y) = (2, -3) + t(2, 0), t \in \mathbb{R}, \begin{cases} x = 2 + 2t \\ y = -3 \end{cases} t \in \mathbb{R}, \text{ no symmetric form,}$
 $y + 3 = 0, y = -3$
- e) $(x, y) = (3, -4) + t(5, 1), t \in \mathbb{R}, \begin{cases} x = 3 + 5t \\ y = -4 + t \end{cases} t \in \mathbb{R}, \frac{x-3}{5} = y + 4,$
 $x - 5y - 23 = 0, y = \frac{1}{5}x - \frac{23}{5}$
- f) $(x, y) = (1, 1) + t(1, 3), t \in \mathbb{R}, \begin{cases} x = 1 + t \\ y = 1 + 3t \end{cases} t \in \mathbb{R}, x - 1 = \frac{y-1}{3},$
 $3x - y - 2 = 0, y = 3x - 2$
- g) $(x, y) = (2, 4) + t(2, -5), t \in \mathbb{R}, \begin{cases} x = 2 + 2t \\ y = 4 - 5t \end{cases} t \in \mathbb{R}, \frac{x-2}{2} = \frac{4-y}{5},$
 $5x + 2y - 18 = 0, y = -\frac{5}{2}x + 9$
- h) $(x, y) = (3, -2) + t(5, 2), t \in \mathbb{R}, \begin{cases} x = 3 + 5t \\ y = -2 + 2t \end{cases} t \in \mathbb{R}, \frac{x-3}{5} = \frac{y+2}{2},$
 $2x - 5y - 16 = 0, y = \frac{2}{5}x - \frac{16}{5}$
- i) $(x, y) = (-1, -4) + t(2, 5), t \in \mathbb{R}, \begin{cases} x = -1 + 2t \\ y = 4 - 3t \end{cases} t \in \mathbb{R}, \frac{x+1}{2} = \frac{4-y}{3},$
 $3x + 2y - 5 = 0, y = -\frac{3}{2}x + \frac{5}{2}$
- j) $(x, y) = (2, 0) + t(-1, 1), t \in \mathbb{R}, \begin{cases} x = 2 - t \\ y = t \end{cases} t \in \mathbb{R}, 2 - x = y,$
 $x + y - 2 = 0, y = -x + 2$
2. a) $90^\circ, 90^\circ, 26.6^\circ, 0^\circ$
 b) $L_2 // L_3 \quad L_1 \perp L_2 \quad L_1 \perp L_3$
 c) $(\frac{19}{5}, \frac{-8}{5}), (\frac{19}{5}, \frac{-8}{5}), (-1, -4)$

$$d) d(P, L_1) = \frac{\sqrt{5}}{5} \quad d(P, L_2) = \sqrt{5} \quad d(P, L_3) = \sqrt{5} \quad d(P, L_4) = 1$$

e) 0

f) A: none B: L_2 C: L_4

$$3. \text{ a) } 3 \quad \text{b) } -3 \quad \text{c) } 0 \quad \text{d) } \mathbb{R} \setminus \{-3, 0, 3\}$$

4. We have $P(0,0)$ and $\vec{n} = (a,b)$. If we take $R(0, \frac{-c}{b})$, then $\vec{PR} = (0, \frac{-c}{b})$. Thus

$$d = \frac{|\vec{PR} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|(0, \frac{-c}{b}) \cdot (a,b)|}{\|(a,b)\|} = \frac{|c|}{\sqrt{a^2 + b^2}}$$

5. We have $\vec{n} = (a,b)$. If we take $R(0, \frac{-c}{b})$, then $\vec{PR} = (-x_0, -y_0 - \frac{c}{b})$. Thus

$$d = \frac{|\vec{PR} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|(-x_0, -y_0 - \frac{c}{b}) \cdot (a,b)|}{\|(a,b)\|} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

6. We have $\vec{n} = (a,b)$. If we take $P_1(0, -\frac{c_1}{b})$ and $P_2(0, \frac{-c_2}{b})$, then $\vec{P_1P_2} = (0, \frac{-c_2 + c_1}{b})$. Thus

$$d = \frac{|\vec{PR} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|(0, \frac{-c_2 + c_1}{b}) \cdot (a,b)|}{\|(a,b)\|} = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

$$7. 3x - 4y - 8 = 0 \quad \text{or} \quad 3x - 4y + 22 = 0$$