MATHEMATICS 201-105-RE
Linear Algebra
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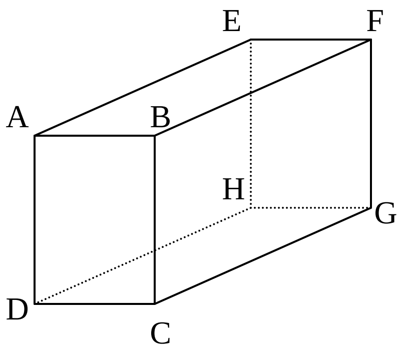
VII - Geometric Vectors

1. Find all vectors in the following parallelepiped that are equivalent to the given vectors.



a) $\overrightarrow{AB}$ b) $\overrightarrow{HE}$ c) $\overrightarrow{HA}$
d) $\overrightarrow{AB}+\overrightarrow{AE}$ e) $\overrightarrow{AB}+\overrightarrow{BC}$ f) $\overrightarrow{CD}+\overrightarrow{BF}$
g) $\overrightarrow{AD}+\overrightarrow{HE}$ h) $\overrightarrow{FA}+\overrightarrow{BG}$ i) $\overrightarrow{FG}-\overrightarrow{FE}$
j) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CE}$ k) $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{AE}$ l) $\overrightarrow{BC}-\overrightarrow{DC}-\overrightarrow{BD}$
m) $\overrightarrow{HC}-\overrightarrow{HA}-\overrightarrow{EC}$ n) $\overrightarrow{GH}+\overrightarrow{BE}-\overrightarrow{CE}-\overrightarrow{FA}$ o) $\overrightarrow{AG}+\overrightarrow{CB}+\overrightarrow{EC}+\overrightarrow{GA}$

2. Let ABCDEF be a regular hexagon where $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{FA}=\vec{b}$.
   a) Express the other sides, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$ and $\overrightarrow{EF}$, in terms of $\vec{a}$ and $\vec{b}$.
   b) Express $\overrightarrow{FB}$, $\overrightarrow{FC}$ and $\overrightarrow{FD}$ in terms of $\vec{a}$ and $\vec{b}$.

3. Consider the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ such that

$$\|\vec{u}\|=4,\ \text{N}15°\text{E} \qquad \|\vec{v}\|=3,\ \text{S}45°\text{W} \qquad \|\vec{w}\|=6,\ \text{N}60°\text{W}$$

   Find the length and direction of the following vectors.
   a) $\vec{u}+\vec{v}$ b) $\vec{u}+\vec{w}$ c) $\vec{w}-\vec{u}$
   d) $\frac{1}{2}\vec{u}-\frac{1}{3}\vec{v}$ e) $\vec{u}-\vec{w}+\vec{v}$ f) $\dfrac{2\vec{v}-3\vec{w}}{5}$

4. An airplane has a maximum air speed of 500 km/h. If the plane is flying at is maximum speed with a heading of 40 degrees west of north and the wind is blowing from north to south at 40 km/h, find the ground speed of the aircraft and the direction.

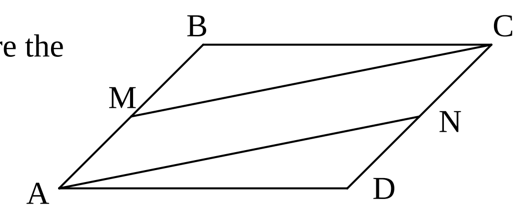
5. A jet if flying through a wind that is blowing with a sped of 40 km/h in the direction N30°E. The jet has a speed of 610 km/h in still air and the pilot head the jet in the direction N45°E. Find the speed and direction of the jet.

6. A woman launches a boat from on shore of a straight river and wants to land at the point directly on the opposite shore. If the speed of the boat (in still water) is 10 km/h and the river is flowing east at the rate of 5 km/h, in what direction should she head the boat in order to arrive at the desired landing point?

7. A boat heads in the direction $\text{N}72°\text{E}$. The speed of the boat in still water is 24 km/h. The water is flowing directly south. It is observed that the true direction of the boat is directly east. Find the speed of the water and the true speed of the boat.

8. Prove that if $\overrightarrow{AB} = \overrightarrow{CD}$ then $\overrightarrow{AC} = \overrightarrow{BD}$.

9. In the following parallelogram ABCD, M and N are the midpoints of AB and CD respectively. Prove that AMCN is a parallelogram.



10. Prove that if the midpoints of the adjacent sides in a rectangle are joined, the resulting figure is a rhombus (a parallelogram whose sides all have equal length).

11. Let ABCD be a parallelogram. Verify that the diagonal BD and the line AE, where E is the midpoint of BC, intersect at a point that divides both of these segments in the ratio of 2 to 1.

12. Prove that the line segment joining the midpoints of the diagonals in a trapezoid is parallel to the base and is half the length of the difference between the lengths of the two bases.

13. Let ABCD be a parallelogram and let E divide the segment AB in a ratio of 2 to 1 and let F divide the segment DC in two. In what ratio does P, the point of intersection of AF and DE, divide the segments AF and DE?

14. Let ABC be a triangle and let D divide the segment AB in a ratio of 3 to 2 and let E divide the segment BC in a ratio of 3 to 4. In what ratio does P, the point of intersection of AE and CD, divide the segments AE and CD?

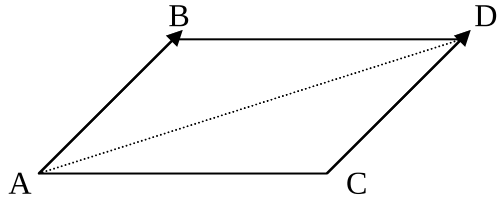
15. Determine whether the following statements are true of false. Explain
    a) Two equivalent vectors have the same initial point.
    b) $2\overrightarrow{AA} = -5\overrightarrow{AA}$
    c) If $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{AD}$ then $\left\|\overrightarrow{AB}\right\| = \left\|\overrightarrow{AC}\right\| + \left\|\overrightarrow{AD}\right\|$.
    d) $2\overrightarrow{AB}$ and $-3\overrightarrow{AB}$ have the same direction.
    e) $2\overrightarrow{AB}$ and $-5\overrightarrow{BA}$ have the same direction.
    f) $\vec{u}$ and $k\vec{u}$ have the same direction.
    g) If $\vec{u} = k\vec{v}$ then $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$.
    h) There exists two nonzero vectors $\vec{u}$ and $\vec{v}$ such that $\|\vec{u} - \vec{v}\| = \|\vec{u}\| - \|\vec{v}\|$.

i) $\left\|\vec{u}-\left(\vec{u}+\vec{v}\right)\right\|=\left\|\vec{v}\right\|$

j) $\left\|\frac{\vec{v}}{\left\|\vec{v}\right\|}\right\|=1$

Answers

1. a) $\overrightarrow{EF}=\overrightarrow{DC}=\overrightarrow{HG}$ b) $\overrightarrow{GF}=\overrightarrow{DA}=\overrightarrow{CB}$ c) $\overrightarrow{GB}$
   d) $\overrightarrow{AF}=\overrightarrow{DG}$ e) $\overrightarrow{AC}=\overrightarrow{EG}$ f) $\overrightarrow{CH}=\overrightarrow{BE}$
   g) $\vec{0}$ h) $\overrightarrow{FH}=\overrightarrow{BD}$ i) $\overrightarrow{EG}=\overrightarrow{AC}$
   j) $\overrightarrow{AE}=\overrightarrow{BF}=\overrightarrow{DH}=\overrightarrow{CG}$ k) $\overrightarrow{AG}$ l) $\vec{0}$
   m) $\overrightarrow{AE}=\overrightarrow{BF}=\overrightarrow{DH}=\overrightarrow{CG}$ n) $\overrightarrow{BG}=\overrightarrow{AH}$ o) $\overrightarrow{EB}=\overrightarrow{HC}$
2. a) $\overrightarrow{BC}=\vec{a}-\vec{b}$ $\overrightarrow{CD}=-\vec{b}$ $\overrightarrow{DE}=-\vec{a}$ $\overrightarrow{EF}=\vec{b}-\vec{a}$
   b) $\overrightarrow{FB}=\vec{a}+\vec{b}$ $\overrightarrow{FC}=2\vec{a}$ $\overrightarrow{BD}=2\vec{a}-\vec{b}$
3. a) $\|\vec{u}+\vec{v}\|=2.0$, N32°W b) $\|\vec{u}+\vec{w}\|=8.0$, N31°W c) $\|\vec{w}-\vec{u}\|=6.3$, S82°W
   d) $\left\|\frac{1}{2}\vec{u}-\frac{1}{3}\vec{v}\right\|=2.9$, N25°E e) $\|\vec{u}-\vec{w}+\vec{v}\|=4.3$, S73°E f) $\left\|\dfrac{2\vec{v}-3\vec{w}}{5}\right\|=3.5$, S41°E
4. 470 km/h N43°W
5. 648.7 km/h N44.1°E
6. N30°W
7. 7.4 km/h and 22.8 km/h
8.



$\overrightarrow{AD}=\overrightarrow{AC}+\overrightarrow{CD}$

$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$

Thus $\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AB}+\overrightarrow{BD}$

$\overrightarrow{AC}+\overrightarrow{AB}=\overrightarrow{AB}+\overrightarrow{BD}$ Since $\overrightarrow{AB}=\overrightarrow{CD}$

$\overrightarrow{AC}=\overrightarrow{BD}$

9. We need to show that $\overrightarrow{AM}=\overrightarrow{NC}$ and $\overrightarrow{AN}=\overrightarrow{MC}$.

For $\overrightarrow{AM}=\overrightarrow{NC}$, $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AB}$ since M is the midpoint of AB

$=\frac{1}{2}\overrightarrow{DC}$ since ABCD is a parallelogram

$=\overrightarrow{NC}$ since N is the midpoint of CD

For $\overrightarrow{AN}=\overrightarrow{MC}$, we have $\overrightarrow{MC}=\overrightarrow{MB}+\overrightarrow{BC}$

$=\overrightarrow{AM}+\overrightarrow{BC}$ since M is the midpoint of AB

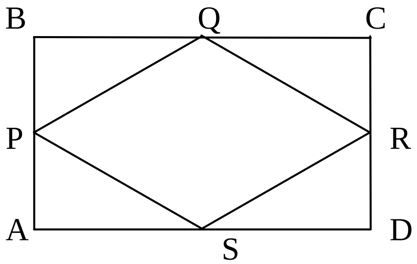
$=\overrightarrow{NC}+\overrightarrow{AD}$ since $\overrightarrow{AM}=\overrightarrow{NC}$ and ABCD is a parallelogram

$=\overrightarrow{DN}+\overrightarrow{AD}$ since N is the midpoint of CD

$=\overrightarrow{AD}+\overrightarrow{DN}$

$=\overrightarrow{AN}$

10. We need to show that PQRS is a parallelogram whose sides have equal length.



Since ABCD is a rectangle and P,Q,R and S are midpoints, then

$$\overrightarrow{AP} = \overrightarrow{PB} = \overrightarrow{DR} = \overrightarrow{RC}$$
$$\overrightarrow{AS} = \overrightarrow{SD} = \overrightarrow{BQ} = \overrightarrow{QC}$$

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{AS} + \overrightarrow{SR} + \overrightarrow{RC} + \overrightarrow{CQ}\\ &= \overrightarrow{PA} + \overrightarrow{BQ} + \overrightarrow{SR} + \overrightarrow{AP} + \overrightarrow{QB}\\ &= -\overrightarrow{AP} - \overrightarrow{QB} + \overrightarrow{SR} + \overrightarrow{AP} + \overrightarrow{QB}\\ &= \overrightarrow{SR}\end{aligned}$$

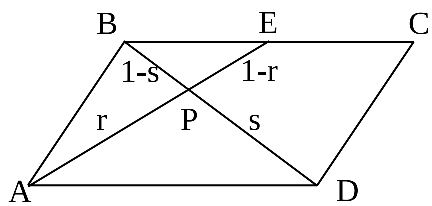
$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QP} + \overrightarrow{PS} + \overrightarrow{SR}\\ &= -\overrightarrow{PQ} + \overrightarrow{PS} + \overrightarrow{SR}\\ &= -\overrightarrow{SR} + \overrightarrow{PS} + \overrightarrow{SR}\\ &= \overrightarrow{PS}\end{aligned}$$

To show that all sides are of equal length, it suffices to show that $\left\|\overrightarrow{PQ}\right\| = \left\|\overrightarrow{PS}\right\|$ since we have proved that PQRS is a parallelogram.

$$\begin{aligned}\left\|\overrightarrow{PQ}\right\|^2 &= \left\|\overrightarrow{PB}\right\|^2 + \left\|\overrightarrow{BQ}\right\|^2 && \text{(by the Pythagorean theorem)}\\ &= \left\|\overrightarrow{AP}\right\|^2 + \left\|\overrightarrow{AS}\right\|^2\\ &= \left\|\overrightarrow{PS}\right\|^2 && \text{(by the Pythagorean theorem)}\end{aligned}$$

hence $\left\|\overrightarrow{PQ}\right\| = \left\|\overrightarrow{PS}\right\|$

11. We need to show that if $\overrightarrow{AP} = r\overrightarrow{AE}$ then $r = \frac{2}{3}$ and if $\overrightarrow{DP} = s\overrightarrow{DB}$ then $s = \frac{2}{3}$.



Since ABCD is a parallelogram and E is the midpoint of BC, then

$$\overrightarrow{BE} = \tfrac{1}{2}\overrightarrow{BC} = \tfrac{1}{2}\overrightarrow{AD}$$

Let us express $\overrightarrow{AP}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in two different ways.

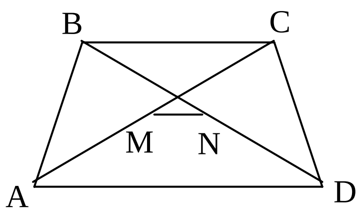
$$\begin{aligned}\overrightarrow{AP} &= r\overrightarrow{AE}\\ &= r\left(\overrightarrow{AB} + \overrightarrow{BE}\right)\\ &= r\left(\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}\right)\\ &= r\overrightarrow{AB} + \tfrac{r}{2}\overrightarrow{BC}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{AD} + s\overrightarrow{DB}\\ &= \overrightarrow{AD} + s\left(\overrightarrow{DA} + \overrightarrow{AB}\right)\\ &= \overrightarrow{AD} + s\left(-\overrightarrow{AD} + \overrightarrow{AB}\right)\\ &= (1-s)\overrightarrow{AD} + s\overrightarrow{AB}\\ &= (1-s)\overrightarrow{BC} + s\overrightarrow{AB}\end{aligned}$$

By the basis theorem, we have the equations $r = s$, $\frac{r}{2} = 1 - s$.

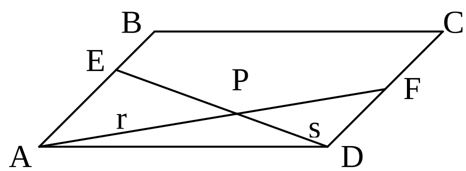
Solving, we obtain $r = \frac{2}{3}$ and $s = \frac{2}{3}$.

12. We need to show that $\overrightarrow{MN}=\frac{1}{2}\left(\overrightarrow{AD}-\overrightarrow{BC}\right)$



$$\begin{aligned}\overrightarrow{MN}&=\overrightarrow{MA}+\overrightarrow{AD}+\overrightarrow{DN}\\&=\tfrac{1}{2}\overrightarrow{CA}+\overrightarrow{AD}+\tfrac{1}{2}\overrightarrow{DB}\qquad\text{since M and N are the midpoints of AC and DB}\\&=\tfrac{1}{2}\overrightarrow{CB}+\tfrac{1}{2}\overrightarrow{BA}+\overrightarrow{AD}+\tfrac{1}{2}\overrightarrow{DC}+\tfrac{1}{2}\overrightarrow{CB}\\&=\tfrac{1}{2}\overrightarrow{CB}+\overrightarrow{AD}+\tfrac{1}{2}\left(\overrightarrow{DC}+\overrightarrow{CB}+\overrightarrow{BA}\right)\\&=-\tfrac{1}{2}\overrightarrow{BC}+\overrightarrow{AD}+\tfrac{1}{2}\overrightarrow{DA}\\&=\overrightarrow{AD}-\tfrac{1}{2}\overrightarrow{AD}-\tfrac{1}{2}\overrightarrow{BC}\\&=\tfrac{1}{2}\overrightarrow{AD}-\tfrac{1}{2}\overrightarrow{BC}\end{aligned}$$

13. If $\overrightarrow{AP}=r\overrightarrow{AF}$ and $\overrightarrow{DP}=s\overrightarrow{DE}$, then we need to find *r* and *s*.



Since ABCD is a parallelogram, we have

$$\overrightarrow{AB}=\overrightarrow{DC}\text{ and }\overrightarrow{AD}=\overrightarrow{BC}.$$

Also, by the definitions of E and F, we have

$$\overrightarrow{AE}=\tfrac{2}{3}\overrightarrow{AB}\text{ and }\overrightarrow{DF}=\tfrac{1}{2}\overrightarrow{DC}.$$

Let us express $\overrightarrow{AP}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$ in two different ways.

$$\begin{aligned}\overrightarrow{AP}&=r\overrightarrow{AF}\\&=r\left(\overrightarrow{AD}+\overrightarrow{DF}\right)\\&=r\overrightarrow{AD}+\tfrac{1}{2}r\overrightarrow{DC}\\&=r\overrightarrow{AD}+\tfrac{r}{2}\overrightarrow{AB}\end{aligned}$$

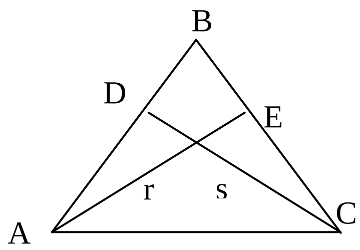
$$\begin{aligned}\overrightarrow{AP}&=\overrightarrow{AD}+\overrightarrow{DP}\\&=\overrightarrow{AD}+s\overrightarrow{DE}\\&=\overrightarrow{AD}+s\left(\overrightarrow{DA}+\overrightarrow{AE}\right)\\&=\overrightarrow{AD}+s\left(-\overrightarrow{AD}+\tfrac{2}{3}\overrightarrow{AB}\right)\\&=(1-s)\overrightarrow{AD}+\tfrac{2s}{3}\overrightarrow{AB}\end{aligned}$$

By the basis theorem, we have the equations

$$r=1-s$$
$$\tfrac{r}{2}=\tfrac{2s}{3}$$

Solving, we obtain $r=\frac{4}{7}$ and $s=\frac{3}{7}$. Hence, P divides AF in a ration of 4 to 3 and divides DE in a ration of 3 to 4.

14. If $\overrightarrow{AP} = r\overrightarrow{AF}$ and $\overrightarrow{DP} = s\overrightarrow{DE}$, then we need to find *r* and *s*.



By the definitions of D and E, we have

$$\overrightarrow{AD} = \tfrac{3}{5}\overrightarrow{AB} \text{ and } \overrightarrow{CE} = \tfrac{4}{7}\overrightarrow{CB}.$$

Let us express $\overrightarrow{AP}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in two different ways.

$$\begin{aligned}\overrightarrow{AP} &= r\overrightarrow{AE} \\ &= r\left(\overrightarrow{AB} + \overrightarrow{BE}\right) \\ &= r\overrightarrow{AB} + \tfrac{3}{7}r\overrightarrow{BC} \\ &= r\overrightarrow{AB} + \tfrac{3}{7}r\left(\overrightarrow{BA} + \overrightarrow{AC}\right) \\ &= \tfrac{4}{7}r\overrightarrow{AB} + \tfrac{3}{7}r\overrightarrow{AC}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{AC} + \overrightarrow{CP} \\ &= \overrightarrow{AC} + s\overrightarrow{CD} \\ &= \overrightarrow{AD} + s\left(\overrightarrow{CA} + \overrightarrow{AD}\right) \\ &= \overrightarrow{AD} + s\left(-\overrightarrow{AC} + \tfrac{3}{5}\overrightarrow{AB}\right) \\ &= (1-s)\overrightarrow{AD} + \tfrac{3s}{5}\overrightarrow{AB}\end{aligned}$$

By the basis theorem, we have the equations

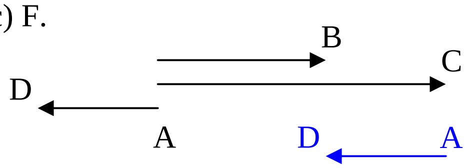
$$\tfrac{4}{7}r = \tfrac{3}{5}s$$
$$\tfrac{3}{7}r = 1 - s$$

Solving, we obtain $r = \tfrac{21}{29}$ and $s = \tfrac{20}{29}$. Hence, P divides AE in a ration of 21 to 8 and divides CD in a ration of 20 to 9.

15. a) F. They may have any initial point as long as the have the same directions and magnitude.

b) T. $\overrightarrow{AA} = \vec{0}$ so $\vec{0} = \vec{0}$.

c) F.



If $\overrightarrow{AC}$, and $\overrightarrow{AD}$ have opposite direction then $\left\|\overrightarrow{AB}\right\| = \left\|\overrightarrow{AC}\right\| - \left\|\overrightarrow{AD}\right\|$.

d) F. They have opposite direction.

e) T. $\overrightarrow{BA} = -\overrightarrow{AB}$.

f) F. If $k = -1$ then $\vec{u} = -\vec{v}$ and we have $\|\vec{u} + \vec{v}\| = \|-\vec{v} + \vec{v}\| = \|\vec{0}\| = 0$

$$\|\vec{u}\| + \|\vec{v}\| = \|-\vec{v}\| + \|\vec{v}\| = |-1| \cdot \|\vec{v}\| + \|\vec{v}\| = 2\|\vec{v}\|$$

g) T. If $\vec{u} = 2\vec{v}$ then $\|\vec{u} - \vec{v}\| = \|2\vec{v} - \vec{v}\| = \|\vec{v}\|$ and $\|\vec{u}\| - \|\vec{v}\| = \|2\vec{v}\| - \|\vec{v}\| = 2\|\vec{v}\| - \|\vec{v}\| = \|\vec{v}\|$

h) T. $\|\vec{u} - (\vec{u} + \vec{v})\| = \|\vec{u} - \vec{u} - \vec{v}\| = |-1| \cdot \|\vec{v}\| = \|\vec{v}\|$

i) T. $\left\|\dfrac{\vec{v}}{\|\vec{v}\|}\right\| = \dfrac{1}{\|\vec{v}\|}\|\vec{v}\| = 1$

j) F. If $k = -1$ then $\vec{u}$ and $k\vec{u} = -\vec{u}$ have opposite direction.