MATHEMATICS 201-105-RE
Linear Algebra
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V - Adjoints and Cramer's Rule

1. Find the adjoint of the matrix A. Then use the adjoint to find the inverse of A, if possible.

   a) $A=\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}$

   b) $A=\begin{bmatrix}-2 & 5\\ 3 & -10\end{bmatrix}$

   c) $A=\begin{bmatrix}2 & -1 & 4\\ 6 & 3 & -2\\ 0 & 3 & 1\end{bmatrix}$

   d) $A=\begin{bmatrix}2 & 1 & 3\\ 0 & 5 & 2\\ 0 & -10 & -4\end{bmatrix}$

   e) $A=\begin{bmatrix}2 & -1 & 4\\ 0 & 3 & 5\\ 0 & 0 & 1\end{bmatrix}$

   f) $A=\begin{bmatrix}0 & 2 & 0 & 1\\ 3 & -1 & 4 & 0\\ 0 & 0 & 1 & 2\\ -1 & 1 & 0 & 2\end{bmatrix}$

2. Use Cramer's Rule to solve the given system of linear equations, if possible.

   a) $\begin{aligned}3x-2y&=2\\ x+4y&=7\end{aligned}$

   b) $\begin{aligned}4x-y-z&=1\\ 2x+2y+3z&=10\\ 5x-2y-2z&=-1\end{aligned}$

   c) $\begin{aligned}3x+3y+5z&=1\\ 3x+5y+9z&=2\\ 5x+9y+17z&=4\end{aligned}$

3. Use Cramer's Rule to solve for x_2.

   a) $\begin{aligned}2x_1-3x_2+x_3&=3\\ x_1+x_2-x_3&=-1\\ 3x_3-2x_2+5x_3&=8\end{aligned}$

   b) $\begin{aligned}x_1-x_2+3x_3-x_4&=2\\ 5x_2+2x_4&=1\\ 2x_1-2x_3-x_4&=5\\ x_1+x_2+4x_3&=9\end{aligned}$

4. Use the adjoint to find the inverse of $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ if $ad-bc\neq 0$.

5. Find the inverse of $A=\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}$ using the adjoint. (The determinant of this matrix was calculated in the previous exercise sheet.)

6. Prove that if A is an invertible matrix of order $n>1$ then $\det(\operatorname{Adj}A)=(\det A)^{n-1}$.

7. Find the inverse of $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ using the adjoint (assuming $adf \neq 0$). From your result, what can we say about the inverse of a triangular matrix with nonzero entries in the main diagonal?

Answers

1. a) $\mathrm{Adj}(A) = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$ b) $\mathrm{Adj}(A) = \begin{bmatrix} -10 & -5 \\ -3 & -2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -2 & -1 \\ \frac{-3}{5} & \frac{-2}{5} \end{bmatrix}$

c) $\mathrm{Adj}(A) = \begin{bmatrix} 9 & 13 & -10 \\ -6 & 2 & 28 \\ 18 & -6 & 12 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{3}{32} & \frac{13}{96} & \frac{-5}{48} \\ \frac{-1}{16} & \frac{1}{48} & \frac{7}{24} \\ \frac{3}{16} & \frac{-1}{16} & \frac{1}{8} \end{bmatrix}$ d) $\mathrm{Adj}(A) = \begin{bmatrix} 0 & -26 & -13 \\ 0 & -8 & -4 \\ 0 & 20 & 10 \end{bmatrix}$, A^{-1} does not exist

e) $\mathrm{Adj}(A) = \begin{bmatrix} 3 & 1 & -17 \\ 0 & 2 & -10 \\ 0 & 0 & 6 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{-17}{6} \\ 0 & \frac{1}{3} & \frac{-5}{3} \\ 0 & 0 & 1 \end{bmatrix}$

f) $\mathrm{Adj}(A) = \begin{bmatrix} 6 & -3 & 12 & -15 \\ 2 & 1 & -4 & 3 \\ -4 & 4 & -10 & 12 \\ 2 & -2 & 8 & -6 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & \frac{-1}{2} & 2 & \frac{-5}{2} \\ \frac{1}{3} & \frac{1}{6} & \frac{-2}{3} & \frac{1}{2} \\ \frac{-2}{3} & \frac{2}{3} & \frac{-5}{3} & 2 \\ \frac{1}{3} & \frac{-1}{3} & \frac{4}{3} & -1 \end{bmatrix}$

2. a) $x = \frac{11}{7},\ y = \frac{19}{14}$ b) $x = 1,\ y = 1,\ z = 2$ c) $x = 0,\ y = -\frac{1}{2},\ z = \frac{1}{2}$

3. a) $x_2 = -\frac{7}{25}$ b) $x_2 = -\frac{35}{3}$

4. $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

5. $A^{-1} = \begin{bmatrix} \frac{bc}{(b-a)(c-a)} & \frac{-ac}{(b-a)(c-b)} & \frac{ab}{(c-a)(c-b)} \\ \frac{-(b+c)}{(b-a)(c-a)} & \frac{a+c}{(b-a)(c-b)} & \frac{-(a+b)}{(c-a)(c-b)} \\ \frac{1}{(b-a)(c-a)} & \frac{-1}{(b-a)(c-b)} & \frac{1}{(c-a)(c-b)} \end{bmatrix}$

6. We have $A \bullet \mathrm{Adj}(A) = \det(A) I$. Thus

$$\det\left(A \bullet \mathrm{Adj}(A)\right) = \det\left(\det(A) I\right)$$
$$\det(A)\det\left(\mathrm{Adj}(A)\right) = \left(\det(A)\right)^n \det(I)$$
$$\det\left(\mathrm{Adj}(A)\right) = \left(\det(A)\right)^{n-1}$$

7. $A^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{-b}{ad} & \frac{be-cd}{adf} \\ 0 & \frac{1}{d} & \frac{-e}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$ Hence, the inverse of an upper triangular matrix is also an upper triangular matrix, if the inverse exists.