MATHEMATICS 201-105-77
Linear Algebra
Martin Huard
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TEST #4
SOLUTIONS

Question 1 (5 points)

Rather than use the standard definitions of addition and scalar multiplication in $\mathbb{R}^2$, suppose that we define these two operations as follows.

$$(u_1,u_2)\oplus(v_1,v_2)=(u_1+v_1,u_2+v_2)$$
$$k\odot(u_1,u_2)=(k,2k)$$

With these new definitions, is $\mathbb{R}^2$ a vector space? Justify your answer.

No, this is not a vector space.

Axiom 10 fails. If $\vec{u}=(3,4)$, then

$$1\odot\vec{u}=(1,2)\neq(3,4)=\vec{u}$$ (Axiom 5, 8, 9 also fail)

Question 2 (13 points)

Let $W=\{(a,b,a-2b):a,b\in\mathbb{R}\}$.

a) Show that W is a subspace of $\mathbb{R}^3$.

W is nonempty since $\vec{0}=(0,0,0)\in W$

Let $\vec{u}=(a,b,a-2b)$ and $\vec{v}=(c,d,c-2d)$ be in W

1. $\vec{u}+\vec{v}=(a+c,b+d,a-2b+c-2d)=(a+c,b+d,(a+c)-2(b+d))\in W$
2. $k\vec{u}=(ka,kb,k(a-2b))=(ka,kb,ka-2kb)\in W$

Hence W is a subspace.

b) Find a basis for W.

Since $(a,b,a-2b)=a(1,0,1)+b(0,1,-2)$

Then if $S=\{(1,0,1),(0,1,-2)\}$, then $W=span(S)$.

Since S is linearly independent (the two vectors are nonparallel) then S is a basis for W.

c) What is the dimension of W? $\dim(W)=2$

d) Find the coordinate vector of $\vec{u}=(3,-2,7)$ relative to the basis found in (b).

$c_1(1,0,1)+c_2(0,1,-2)=(3,-2,7)$

$$\left[\begin{array}{cc|c}1&0&3\\0&1&-2\\1&-2&7\end{array}\right]\xrightarrow{R_3\to R_3-R_1}\left[\begin{array}{cc|c}1&0&3\\0&1&-2\\0&-2&4\end{array}\right]\xrightarrow{R_3\to R_3+2R_2}\left[\begin{array}{cc|c}1&0&3\\0&1&-2\\0&0&0\end{array}\right]\quad\begin{array}{l}c_2=3\\c_1=-2\end{array}$$

$(\vec{u})_S=(3,-2)$

Question 3 (5 points)

Consider the set $S = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \right\}$. Does S span $M_{2,2}$?

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2}$

$$c_1 \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 2 & 3 & 1 & b \\ 3 & 3 & 4 & c \\ 1 & 1 & 2 & d \end{array}\right] \xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -1 & b-2a \\ 0 & 0 & 1 & c-3a \\ 0 & 0 & 1 & d-a \end{array}\right] \xrightarrow{R_4 \to R_4 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -1 & b-2a \\ 0 & 0 & 1 & c-3a \\ 0 & 0 & 0 & 2a-c+d \end{array}\right]$$

Since there are is no solution if $2a - c + d \neq 0$, then S does not span $M_{2,2}$.

Question 4 (10 points)

Are the following sets W bases for the vector space V? Support your answer.

a) $S = \{(1,2,3),(-3,-5,-9),(2,8,7)\}$, $V = \mathbb{R}^3$

$$c_1(1,2,3) + c_2(-3,-5,-9) + c_3(2,8,7) = (0,0,0)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -5 & 8 & 0 \\ 3 & -9 & 7 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \begin{aligned} c_3 &= 0 \\ c_2 &= 0 \\ c_1 &= 0 \end{aligned}$$

Since S is linearly independent and $n(S) = 3 = \dim(\mathbb{R}^3)$, then S is a basis for $\mathbb{R}^3$

b) $W = \{1+2x, x+2x^2, 1-4x^2\}$, $V = P_2$

$$c_1(1+2x) + c_2(x+2x^2) + c_3(1-4x^2) = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} c_3 &= t \\ c_2 &= 2t \\ c_1 &= -t \end{aligned}$$

Since W is linearly dependent, then W is not a basis for P_2.

Question 5 (10 points)

Consider the set $S = \{(1,3,0),(2,7,-2),(2,-1,14)\}$.

a) Find a basis for $Span(S)$.

$c_1(1,3,0)+c_2(2,7,-2)+c_3(2,-1,14)=(0,0,0)$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 3 & 7 & -1 & 0 \\ 0 & -2 & 14 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & -2 & 14 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_3 = t$, $c_2 = 7t$ and $c_1 = -16t$

If $t=1$, then $(2,-1,14)=16(1,3,0)-7(2,7,-2)$

By the +/- theorem, if $S' = \{(1,3,0),(2,7,2)\}$ then $Span(S)=Span(S')$.

Since S' is linearly independent (the two vectors are not multiples of each other) then S' is a basis for $Span(S)$.

b) What is the dimension for $Span(S)$? $\dim(Span(S))=2$

c) Give a geometrical description for *S*.

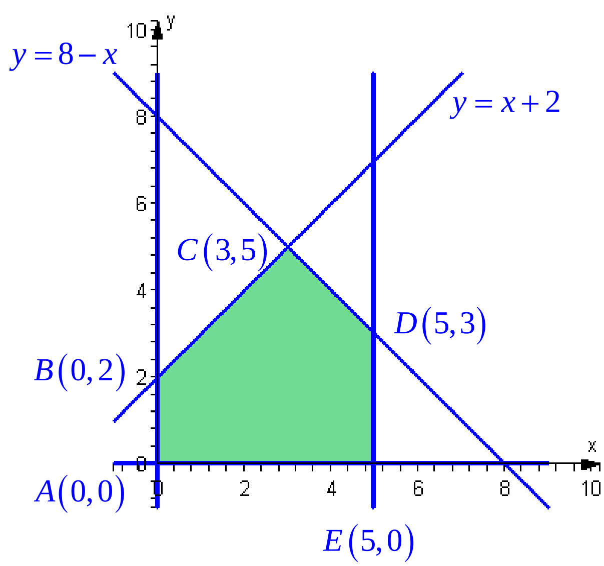
$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 3 & 0 \\ 2 & 7 & -2 \end{vmatrix} = (-6,2,1)$$

$Span(S)$ is the plane $6x-2y-z=0$

Question 6 (7 points)

Maximize $z = 2x+3y$

subject to
$$x+y \le 8$$
$$-x+y \le 2$$
$$x \le 5$$
$$x \ge 0,\ y \ge 0$$



At $A(0,0)$, $z=0$

$B(0,2)$, $z=6$

$C(3,5)$, $z=21$

$D(5,3)$, $z=19$

$E(5,0)$, $z=10$

Thus the maximum value for *z* is 21 when $x=3$ and $y=5$.