MATHEMATICS 201-105-RE
Linear Algebra
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Winter 2006

Review Exercises

1. Consider the matrices $A=\begin{bmatrix}2 & -2 & 3\\ 1 & 5 & -1\\ 0 & 3 & 1\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2 & 2\\ -3 & 4 & 1\end{bmatrix}$ and $C=\begin{bmatrix}2 & -1\\ 1 & 6\end{bmatrix}$. Evaluate, if possible. (For f, g, and h use your answer from (e)).
   a) BA b) B^TB-3A c) CB d) $\operatorname{tr}\left(3BB^T+C\right)$
   e) $\det(A)$ f) $\det\left(A^3\right)$ g) $\det(2A)$ h) $\det\left(AA^T\right)$
   i) $\det\left(\operatorname{adj}(A)\right)$

2. A square matrix A is called **skew-symmetric** if $A^T=-A$.
   a) Prove that if A is invertible and skew-symmetric, then A^{-1} is skew-symmetric.
   b) Prove that A^T, $A+B$ and kA are skew-symmetric if A and B are skew symmetric.
   c) Prove that $A-A^T$ is skew symmetric.

3. Prove that if A and B are $n\times n$ matrices such that $A^2=B^2=(AB)^2=I$ then $AB=BA$.

4. Let A be a matrix such that $A^3=I$.
   a) Prove that A is invertible.
   b) Prove that $(I-2A)^{-1}=I+2A+4A^2$

5. Solve the following systems of linear equations, if possible.
   a) $$\begin{aligned}2x-y+2z&=1\\ x+y-3z&=4\\ 5x-y+z&=6\\ x+4y-11z&=11\end{aligned}$$
   b) $$\begin{aligned}4x-y+2z&=3\\ 2x-5y+z&=9\\ 2x+4y+z&=-6\end{aligned}$$
   c) $$\begin{aligned}2x-y+z-5t&=12\\ 6x+3y-2z+3w+t&=1\\ 2x+5y-4z+3w+11t&=8\end{aligned}$$
   d) $$\begin{aligned}3x-y+2z&=9\\ 5x-y+3z&=16\\ 2x+3y-z&=11\end{aligned}$$

6. For which values of a will the following system of linear equations have

$$\begin{aligned} x - y + 2z &= 7 \\ -2x + ay + -4z &= 5a - 24 \\ 3x + 2y + (6-a)z &= 4 \end{aligned}$$

a) a unique solution
b) no solution
c) an infinite number of solutions

7. Solve each of the following systems of linear equations using
(i) Cramer's Rule
(ii) The inverse.

a)
$$\begin{aligned} 3x - y + &= 1 \\ 5x + 3z &= 14 \\ x + 3y - z &= 4 \end{aligned}$$

b)
$$\begin{aligned} 2x + y + 4z &= 8 \\ 2x - y + &= -16 \\ 3x + 5z &= 2 \end{aligned}$$

8. Consider the matrix $A = \begin{bmatrix} 3 & -1 \\ 5 & 4 \end{bmatrix}$.

a) Find the inverse of A using the adjoint.
b) Express A^{-1} as a product of elementary matrices.
c) Express A as a product of elementary matrices.

9. Consider the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 3 & 0 \end{bmatrix}$.

a) Find the inverse of A using the adjoint.
b) Express A^{-1} and A as a product of elementary matrices.

10. A coin bank has only nickels, dimes and quarters. The value of the coins is \$2. There are twice as many nickles as dimes, and one more dime than quarters. Find the number of each coin in the bank.

11. Suppose a man has three modes of transportation to work: he can walk, drive his car, or take the bus. If he walks one day, then he will either take the car the next day with a probability of $\frac{2}{3}$ or take the bus with a probability of $\frac{1}{3}$. If he drove one day, then he will walk the next day with a probability of $\frac{1}{2}$ and take the bus with a probability of $\frac{1}{2}$. If he took the bus one day, then he will walk the next day with a probability of $\frac{1}{3}$, drive with a probability of $\frac{1}{3}$ and take the bus with a probability of $\frac{1}{3}$.

a) Find the transition matrix.
b) If he drives on Monday, find the probability that he will walk, drive or take the bus on Wednesday.
c) In the long run, how often will he walk, drive and take the bus?

12. Find the equation of the parabola passing through the points $A(1,4)$, $B(2,12)$ and $C(-3,32)$.

13. Let ABC be a triangle and E a point on the segment BC dividing it in a ration of 1 to 3. Let D be the midpoint of AC. Join A to E and B to D, and let P be the point on intersection of the segments AE and BD. In what ration does P divide AE and BD?

14. Let ABC be a triangle and M, N and P the midpoints of AB, BC and CA respectively. Prove that if O is any point (inside or outside the triangle) then
$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}$$

15. A Boeing 737 aircraft maintains a constant airspeed of 500 miles per hour in the direction due south. The velocity of the jet stream is 80 miles per hour in a northeasterly direction (N45^{o}E). Find the actual speed and direction of the aircraft relative to the ground.

16. A river flows from west to est. There are ferry terminals on the north and south shores, the north dock being 15° east of north (i.e.N15°E) from the south dock. The ferry captain knows from experience that in order to reach the dock on the north shore from the south shore dock, she has to steer N30°W.
    a) If the ferry travels at 12 km/h, what is the speed of the current?
    b) If the trip takes ¼ hour, how far apart are the dock?

17. Consider the vectors $\vec{u}=(-2,5,5)$, $\vec{v}=(1,-1,2)$ and $\vec{w}=(5,-1,2)$.
    a) Evaluate $2\vec{u}-3\vec{v}$
    b) Find the vector projection of $\vec{u}$ onto $\vec{w}$.
    c) Find the angle between $\vec{u}$ and $\vec{w}$.
    d) Find the area of the triangle having sides $\vec{u}$ and $\vec{v}$.
    e) Find the volume of the parallelepiped having sides $\vec{u}$, $\vec{v}$ and $\vec{w}$.

18. Consider the points *A*(2,-1,3), and *B*(3,-1,5), *C*(-2,2,3) and *D*(-1,0,5).
    a) Find the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.
    b) Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
    c) Find the volume of the tetrahedron ABCD.
    d) Find the equation of the line (in ***parametric form***) passing through *D* and parallel to $\overrightarrow{AB}$.
    e) Find the equation of the plane (in ***general form***) parallel to $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and passing through *D*.
    f) Find the equation of the plane perpendicular to $\overrightarrow{AC}$ and passing through *D*.

19. Consider the plane $\pi : 2x + y - 5z + 1 = 0$ and the line $L : \frac{x-1}{3} = \frac{2y+1}{4} = 3 - z$

a) Find the equation of the line (in ***symmetric form***) perpendicular to π and passing through P(1,1,-3).
b) Find the equation of the line (in ***parametric form***) parallel to L and passing through P(1,1,-3).
c) Find the distance between the line L and the line found in (b).
d) Find the intersection, if possible, of the plane π and the line L.
e) Find the equation of the plane π in vector form.
f) Find the equation of the plane π_2 (in ***general form***) perpendicular to L and passing through P(1,1,-3).
g) Find the angle between the plane π and the plane π_2 found in (f).
h) Find the point Q on the plane π that is closest to the point P(1,1,-3).
i) Find the point Q on the line L that is closest to the point P(1,1,-3).
j) Find the distance from the point P(1,1,-3) to the plane π.
k) Find the distance from the point P(1,1,-3) to the line L.
l) Find the equation of the plane (if possible), in general form containing the lines L and $L_2 : \frac{x+5}{3} = \frac{y-2}{2} = -z$.
m) Find the equation of the plane (if possible), in general form containing the lines L and $L_3 : \frac{x+4}{2} = y + \frac{5}{2} = \frac{z+10}{3}$.

20. Is the set V a vector space with the following operations? Support your answer.

a) $V = \mathbb{R}^2$ $\quad (u_1, u_2) \oplus (v_1, v_2) = (u_1 - v_1, u_2 - v_2)$

$k \odot (u_1, u_2) = (ku_1, ku_2)$

b) $V = \mathbb{R}^2$ $\quad (u_1, u_2) \oplus (v_1, v_2) = (u_1 + v_1 + 1, u_2 + v_2 + 1)$

$k \odot (u_1, u_2) = (k + ku_1 - 1, k + ku_2 - 1)$

21. Is the set W a subspace of V? Support your answer.

a) $W = \left\{ \begin{bmatrix} a & a+b \\ a-b & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ $\quad V = M_{2,2}$
b) $W = \{ A : A \text{ is nilpotent, } A \in M_{2,2} \}$ $\quad V = M_{2,2}$ $\quad$ (A is ***nilpotent*** if $A^2 = 0$)
c) $W = \{ ax^3 - b : a, b \in \mathbb{R} \}$ $\quad V = P_3$
d) $W = \{ p(x) : p(1) = p(2),\ p(x) \in P_2 \}$ $\quad V = P_2$
e) $W = \{ (x, y, z) : 2x - 3y + z = 0,\ x, y, z \in \mathbb{R} \}$ $\quad V = \mathbb{R}^3$
f) $W = \{ (x, y, z) : 2x - 3y + z - 8 = 0,\ x, y, z \in \mathbb{R} \}$ $\quad V = \mathbb{R}^3$
g) $W = \{ (a, 3a - 4, a + 1) : a \in \mathbb{R} \}$ $\quad V = \mathbb{R}^3$

22. For each of the subsets W in $\mathbb{R}^3$,

i) Find a basis for W.
ii) Find the dimension of W
iii) Give a geometrical interpretation of W.

a) $W=\left\{\left(2a+b,a,a-2b\right):\ a,b\in\mathbb{R}\right\}$

b) $W=\left\{\left(a-2b+3c,3a+b+2c,a+4b-3c\right):\ a,b,c\in\mathbb{R}\right\}$

c) $W=\left\{\left(2a-b+c,a+b+c,3a+2c\right):\ a,b,c\in\mathbb{R}\right\}$

23. Find a basis and the dimension of each of the following subspaces W,

a) $W=\left\{\begin{bmatrix} a & a+b \\ a-b & b \end{bmatrix}: a,b\in\mathbb{R}\right\}$

b) $W=\left\{\begin{bmatrix} a+b+2c & 2a+3b+3c \\ 2a+3b+3c & -a+b-4c \end{bmatrix}: a,b,c\in\mathbb{R}\right\}$

c) $W=\left\{ax^2+\left(b-a\right)x+b:\ a,b\in\mathbb{R}\right\}$

24. Find all values of t for which S is linearly independent.

a) $S=\left\{\left(2,3,5\right),\left(-1,t,-1\right),\left(-1,-1,t\right)\right\}$

b) $S=\left\{3x^2+x+4,2x^2-x,x^2+tx+2t\right\}$

25. Do the following sets S span V?

a) $S=\left\{\left(2,-4,1\right),\left(1,2,-3\right),\left(5,-14,6\right)\right\}$ $\quad V=\mathbb{R}^3$

b) $S=\left\{\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix},\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right\}$ $\quad V=S_{2,2}$ (The set of symmetric 2×2 matrices)

c) $S=\left\{x^3-2x+1,x^3+x^2,x^3+x^2+x+1\right\}$ $\quad V=P_3$

26. Are the following sets S bases for the vector space V?

a) $S=\{(2,-1,3),(1,1,7),(-2,4,1)\}$, $V=\mathbb{R}^3$

b) $S=\{(-3,5,1),(2,-7,12)\}$, $V=\mathbb{R}^3$

c) $S=\{x^2+1,x^2-1,x^2+x+1,x^2-x-1\}$, $V=P_2$?

d) $S=\left\{\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix},\begin{bmatrix} 4 & -1 \\ 4 & 3 \end{bmatrix},\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix},\begin{bmatrix} 5 & 5 \\ 1 & -2 \end{bmatrix}\right\}$, $V=M_{2,2}$?

27. Add or subtract vectors to the set S to so that it forms a basis for $\mathbb{R}^3$

a) $S=\left\{\left(1,3,5\right),\left(2,-1,3\right)\right\}$

b) $S=\left\{\left(1,2,4\right),\left(1,3,9\right),\left(1,0,-6\right)\right\}$

28. Find a basis for $Span(S)$ and give a geometrical description (except for (d)) if.

a) $S=\{(-1,1,-1),(2,1,1),(1,5,1)\}$

b) $S=\{(1,3,-2),(2,6,-4),(-3,-9,6)\}$

c) $S=\{(1,2,-1),(2,3,1),(4,7,-1),(1,1,2)\}$

d) $S=\{(1,2,3,4),(4,3,2,1),(1,1,1,1)\}$.

29. Find the coordinate vector for $\vec{w}$ in the vector space V relative to the basis S.

a) $\vec{w}=(-5,-6,24)$, $V=\mathbb{R}^3$ and $S=\{(2,-1,4),(1,2,4),(-3,-3,4)\}$.

b) $\vec{w}=(12,-7,10)$, $V=\mathbb{R}^3$ and $S=\{(3,-1,5),(1,2,3),(2,-1,-1)\}$.

c) $\vec{w}=(4,-3,5,3)$, $V=\text{Span}(S)$ and $S=\{(3,-1,4,0),(0,1,-2,4),(2,2,1,1)\}$.

d) $\vec{w}=9x^2-x+11$, $V=P_2$ and $S=\{x^2+x, x^2-1, 2x^2-3x+4\}$.

30. Find a basis, and the dimension, for the solution space of $AX=0$.

a)
$$\begin{aligned} x-2y+z-w&=0\\ 3x+y+w&=0\\ 4x+6y-2z+4w&=0 \end{aligned}$$

b)
$$\begin{aligned} x-y+3z&=0\\ 2x+5y+6z&=0\\ x-8y+3z&=0\\ 2x+y+6z&=0 \end{aligned}$$

31. Find the minimum or maximum values of the given objective function, subject to the indicated constraints.

a) Objective function: $f=3x+5y$
Constraints:
$$\begin{aligned} x+y&\geq 2\\ 2x+3y&\leq 12\\ 3x+2y&\leq 12\\ x\geq 0,\ y&\geq 0 \end{aligned}$$

b) Objective function: $f=5x+2y$
Constraints:
$$\begin{aligned} x+y&\leq 10\\ 2x+y&\geq 10\\ x+2y&\geq 10\\ x\geq 0,\ y&\geq 0 \end{aligned}$$

c) Objective function: $z=2x+5y$
Constraints:
$$\begin{aligned} 2x+y&\geq 8\\ -4x+y&\leq 2\\ 2x-3y&\leq 0\\ x\geq 0,\ y&\geq 0 \end{aligned}$$

32. An entrepreneur is having a design group produce at least six samples of a new kind of fastener that he wants to market. It cost \$9.00 to produce each metal fastener and \$4.00 to produce each plastic fastener. He wants to have at least two of each version of the fastener and needs to have all the samples 24 hours from now. It takes 4 hours to produce each metal sample and 2 hours to produce each plastic sample. To minimize the cost of the samples, how many of each kind should the entrepreneur order? What will be the cost of the samples?

For problems with the simplex method, see the last exercise sheet.

ANSWERS

1. a) $\begin{bmatrix} 0 & 18 & -3 \\ -2 & 29 & -12 \end{bmatrix}$ b) $\begin{bmatrix} 4 & -8 & -14 \\ -17 & 5 & 11 \\ -5 & -1 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 3 \\ -19 & 26 & 8 \end{bmatrix}$ d) 113

   e) 27 f) 19683 g) 216 h) 729 i) 729

2. a) If A is skew symmetric, then $A^T = -A$. So $\left(A^{-1}\right)^T = \left(A^T\right)^{-1} \underset{\substack{\uparrow \\ A^T=-A}}{=} (-A)^{-1} = -A^{-1}$

   b) If A and B are skew symmetric, then $A^T = -A$ and $B^T = -B$

   $$\left(A^T\right)^T \underset{\substack{\uparrow \\ A^T=-A}}{=} (-A)^T = -A^T \qquad (A+B)^T = A^T + B^T \underset{\substack{\uparrow \\ A^T=-A \\ B^T=-B}}{=} -A-B = -(A+B)$$

   $$(kA)^T = kA^T \underset{\substack{\uparrow \\ A^T=-A}}{=} k(-A) = -(kA)$$

   c) $\left(A - A^T\right)^T = A^T - \left(A^T\right)^T = A^T - A = -\left(A - A^T\right)$

3. $(AB)^2 = (AB)(AB) = I$, then AB is invertible and $(AB)^{-1} = AB$. Also, since $A^2 = I$ and $B^2 = I$ then A and B are invertible with $A^{-1} = A$ and $B^{-1} = B$.

   Thus $AB = (AB)^{-1} = B^{-1}A^{-1} = BA$

4. a)
   $$\begin{aligned} A^3 &= I \\ \det\left(A^3\right) &= \det(I) \\ \left[\det(A)\right]^3 &= 1 \\ \det(A) &= 1 \end{aligned}$$
   Thus, since $\det(A) \neq 0$, then A is invertible.

   b) To prove: $(I-2A)\left(I+2A+4A^2\right) = I$
   $$\begin{aligned} \text{LS} &= (I-2A)\left(I+2A+4A^2\right) \\ &= I^2 + 2IA + 4IA^2 - 2AI - 4A^2 - 8A^3 \\ &= I + 2A + 4A^2 - 2A - 4A^2 - 8(0) \\ &= I \\ &= RS \end{aligned}$$

5. a) $\left(\frac{5}{3}+\frac{1}{3}t, \frac{7}{3}+\frac{8}{3}t, t\right)$ b) $\left(\frac{1}{3}-\frac{1}{2}t, -\frac{5}{3}, t\right)$ c) No solutions d) $(3,2,1)$

6. a) $a \neq 0, 2$ b) $a = 0$ c) $a = 2$

7. a) $(1,2,3)$ b) $(-6,4,4)$

8. a) $A^{-1} = \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{-5}{17} & \frac{3}{17} \end{bmatrix}$ b) $A^{-1} = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{17} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

   c) $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{17}{3} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix}$

9. a) $A^{-1} = \begin{bmatrix} 2 & -1 & \frac{1}{3} \\ -2 & 1 & 0 \\ -3 & 2 & \frac{-2}{3} \end{bmatrix}$

b) $A^{-1}=\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{2}{3}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$A=\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\4&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{3}{2}\end{bmatrix}\begin{bmatrix}1&0&\frac{1}{2}\\0&1&0\\0&0&1\end{bmatrix}$

10. 10 nickels, 5 dimes and 4 quarters.

11. a) $A=\begin{bmatrix}0&\frac{1}{2}&\frac{1}{3}\\\frac{2}{3}&0&\frac{1}{3}\\\frac{1}{3}&\frac{1}{2}&\frac{1}{3}\end{bmatrix}$ b) Walk: $\frac{1}{6}$ Drive: $\frac{1}{2}$ Bus: $\frac{1}{3}$

c) Walk: $\frac{9}{31}$ Drive: $\frac{10}{31}$ Bus: $\frac{12}{31}$

12. $y=2-x+3x^2$

13. 4 to 1 and 2 to 3

14. $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\left(\overrightarrow{OM}+\overrightarrow{MA}\right)+\left(\overrightarrow{ON}+\overrightarrow{NB}\right)+\left(\overrightarrow{OP}+\overrightarrow{PC}\right)$

$=\overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}+\overrightarrow{MA}+\overrightarrow{NB}+\overrightarrow{PC}$

$=\overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}+\frac{1}{2}\overrightarrow{BA}+\frac{1}{2}\overrightarrow{CB}+\frac{1}{2}\overrightarrow{AC}$

$=\overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}+\frac{1}{2}\left(\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BA}\right)$

$=\overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}+\frac{1}{2}\vec{0}$

$=\overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}$

15. 447 miles per hour S7.3°E

16. a) 8.78 km/h b) 2.69 km

17. a) $(-7,13,4)$ b) $\left(\frac{-5}{6},\frac{1}{6},\frac{-1}{3}\right)$ c) $97.1°$ d) $\frac{3\sqrt{35}}{2}$ e) 60

18. a) $\left(\frac{16}{25},\frac{-12}{25},0\right)$ b) $111°$ c) $\frac{8}{3}$

d) $\begin{cases}x=-1+t\\y=0\\z=5+2t\end{cases}$ e) $6x+8y-3z+21=0$ f) $4x-3y+4=0$

19. a) $\dfrac{x-1}{2}=y-1=\dfrac{z+3}{-5}$ b) $\begin{cases}x=1+3t\\y=1+2t\\z=-3-t\end{cases}$

c) $\frac{3\sqrt{707}}{14}$ d) $\left(\frac{101}{26},\frac{37}{26},\frac{53}{26}\right)$ e) $(x,y,z)=(0,-1,0)+s\left(\frac{-1}{2},1,0\right)+t\left(0,1,\frac{1}{5}\right)$

f) $3x+2y-z-8=0$ g) $50.6°$ h) $\left(\frac{-4}{15},\frac{11}{30},\frac{1}{6}\right)$

i) $\left(\frac{41}{14},\frac{11}{14},\frac{33}{14}\right)$ j) $\frac{19\sqrt{30}}{30}$ k) $\frac{3\sqrt{707}}{14}$

l) $7x-30y-39z+95=0$ m) $7x-11y-z-\frac{19}{2}=0$

20. a) No Axiom 2 fails: Counter example with $\vec{u}=(1,2)$ and $\vec{v}=(3,4)$

$$(1,2)\oplus(3,4)=(1-3,2-4)=(-2,-2)$$
$$(3,4)\oplus(1,2)=(3-1,4-2)=(2,2)$$

Thus $(1,2)\oplus(3,4)\neq(3,4)\oplus(1,2)$, that is $\vec{u}\oplus\vec{v}\neq\vec{v}\oplus\vec{u}$.

b) Yes. *(see solutions for details)*

21. a) Yes b) No c) Yes d) Yes e) Yes f) No g) Yes
(*see solutions for the details*)

22. a) $B=\{(2,1,1),(1,0,-2)\}$ $\dim(W)=2$ The plane $2x-5y+z=0$

b) $B=\{(1,3,1),(-2,1,4)\}$ $\dim(W)=2$ The plane $11x-6y+7z=0$.

c) $B=\{(2,1,3),(-1,1,0),(1,1,2)\}$ $\dim(W)=3$

23. a) $B=\left\{\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}\right\}$ $\dim(W)=2$

b) $B=\left\{\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix},\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}\right\}$ $\dim(W)=2$

c) $B=\{x^2-x,x+1\}$ $\dim(W)=2$

24. a) $t\in\mathbb{R}/\{-3,-1\}$ b) $t\in\mathbb{R}/\{2\}$

25. a) No b) Yes c) No

26. a) Yes b) No c) No d) Yes

27. a) $S_B=\{(1,3,5),(2,-1,3),(1,0,0)\}$ b) $S_B=\{(1,2,4),(1,3,9),(1,0,0)\}$

28. a) S is a basis and $\operatorname{span}(S)=\mathbb{R}^3$ b) $S_B=\{(1,3,-2)\}$, the line $l: x=\frac{y}{3}=\frac{z}{-2}$

c) $B=\{(1,2,-1),(2,3,1)\}$, the plane $\pi: 5x-3y-z=0$ d) $B=\{(1,2,3,4),(4,3,2,1)\}$

29. a) $\vec{w}_S=(1,2,3)$ b) $\vec{w}_S=(3,-1,2)$ c) $\vec{w}_S=(2,1,-1)$ d) $\vec{w}_S=\left(6,\frac{-5}{3},\frac{7}{3}\right)$

30. a) $B_{SS}=\left\{\left(\frac{-1}{7},\frac{3}{7},1,0\right),\left(\frac{-1}{7},\frac{-4}{7},0,1\right)\right\}$ $\dim(SS)=2$ b) $B_{SS}=\{(-3,0,1)\}$ $\dim(SS)=1$

31. a) Max of 20 when $x=0$ and $y=4$ Min of 6 when $x=2$ and $y=0$

b) Max of 50 when $x=0$ and $y=10$ Min of $\frac{74}{3}$ when $x=\frac{10}{3}$ and $y=\frac{10}{3}$

c) No max Min of 16 when $x=3$ and $y=2$

32. 2 metal samples, 4 plastic samples; $34