MATHEMATICS 201-105-RE
Linear Algebra
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Review Exercises
SOLUTIONS

1. Consider the matrices $A=\begin{bmatrix}2 & -2 & 3\\ 1 & 5 & -1\\ 0 & 3 & 1\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2 & 2\\ -3 & 4 & 1\end{bmatrix}$ and $C=\begin{bmatrix}2 & -1\\ 1 & 6\end{bmatrix}$. Evaluate, if possible. (For f, g, and h, use your answer from (e)).

a) $BA=\begin{bmatrix}-1 & 2 & 2\\ -3 & 4 & 1\end{bmatrix}\begin{bmatrix}2 & -2 & 3\\ 1 & 5 & -1\\ 0 & 3 & 1\end{bmatrix}=\begin{bmatrix}0 & 18 & -3\\ -2 & 29 & -12\end{bmatrix}$

b) $B^TB-3A=\begin{bmatrix}-1 & -3\\ 2 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}-1 & 2 & 2\\ -3 & 4 & 1\end{bmatrix}-3\begin{bmatrix}2 & -2 & 3\\ 1 & 5 & -1\\ 0 & 3 & 1\end{bmatrix}$

$=\begin{bmatrix}10 & -14 & -5\\ -14 & 20 & 8\\ -5 & 8 & 5\end{bmatrix}-\begin{bmatrix}6 & -6 & 9\\ 3 & 15 & -3\\ 0 & 9 & 3\end{bmatrix}=\begin{bmatrix}4 & -8 & -14\\ -17 & 5 & 11\\ -5 & -1 & 2\end{bmatrix}$

c) $CB=\begin{bmatrix}2 & -1\\ 1 & 6\end{bmatrix}\begin{bmatrix}-1 & 2 & 2\\ -3 & 4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 3\\ -19 & 26 & 8\end{bmatrix}$

d) $\operatorname{tr}\left(3BB^T+C\right)=\operatorname{tr}\left(3\begin{bmatrix}-1 & 2 & 2\\ -3 & 4 & 1\end{bmatrix}\begin{bmatrix}-1 & -3\\ 2 & 4\\ 2 & 1\end{bmatrix}+\begin{bmatrix}2 & -1\\ 1 & 6\end{bmatrix}\right)$

$=\operatorname{tr}\left(3\begin{bmatrix}9 & 13\\ 13 & 26\end{bmatrix}+\begin{bmatrix}2 & -1\\ 1 & 6\end{bmatrix}\right)$

$=\operatorname{tr}\left(\begin{bmatrix}29 & 38\\ 40 & 84\end{bmatrix}\right)$

$=29+84=113$

e) $\det(A)=\begin{vmatrix}2 & -2 & 3\\ 1 & 5 & -1\\ 0 & 3 & 1\end{vmatrix}=2\begin{vmatrix}5 & -1\\ 3 & 1\end{vmatrix}-\begin{vmatrix}-2 & 3\\ 3 & 1\end{vmatrix}+0\begin{vmatrix}-2 & 3\\ 5 & -1\end{vmatrix}=2\cdot 8-(-11)+0=27$

f) $\det\left(A^3\right)=\left[\det(A)\right]^3=27^3=19683$

g) $\det(2A)=2^3\det(A)=8\cdot 27=216$

h) $\det\left(AA^T\right) = \det(A)\det\left(A^T\right) = \det(A)\det(A) = 27^2 = 729$

i) $\det(\operatorname{adj}(A)) = \det\left(\left[\operatorname{cof}(A)\right]^T\right) = \det(\operatorname{cof}(A))$

$$= \begin{vmatrix} 8 & -1 & 3 \\ 11 & 2 & -6 \\ -13 & 5 & 12 \end{vmatrix}$$

$$= 8\begin{vmatrix} 2 & -6 \\ 5 & 12 \end{vmatrix} + \begin{vmatrix} 11 & -6 \\ -13 & 12 \end{vmatrix} + 3\begin{vmatrix} 11 & 2 \\ -13 & 5 \end{vmatrix}$$

$$= 8 \cdot 54 + 54 + 3 \cdot 81 = 729$$

2. A square matrix A is called **skew-symmetric** if $A^T = -A$.

a) Prove that if A is invertible and skew-symmetric, then A^{-1} is skew-symmetric.

To prove: $\left(A^{-1}\right)^T = -A^{-1}$

$$LS = \left(A^{-1}\right)^T$$
$$= \left(A^T\right)^{-1}$$
$$= (-A)^{-1} \qquad \text{since } A \text{ is skew-symmetric } (A^T = -A)$$
$$= -A^{-1}$$
$$= RS$$

b) Prove that A^T, $A+B$ and kA are skew-symmetric if A and B are skew symmetric.

To prove: $\left(A^T\right)^T = -A^T$

$$LS = \left(A^T\right)^T$$
$$= (-A)^T \qquad \text{since } A \text{ is skew-symmetric}$$
$$= -A^T$$
$$= RS$$

To prove: $(A+B)^T = -(A+B)$

$$LS = (A+B)^T$$
$$= A^T + B^T$$
$$= -A - B \qquad \text{since } A \text{ and } B \text{ are skew-symmetric}$$
$$= -(A+B)$$
$$= RS$$

To prove: $(kA)^T = -(kA)$

$$\begin{aligned} LS &= (kA)^T \\ &= kA^T \\ &= k(-A) \qquad \text{since } A \text{ is skew-symmetric} \\ &= -(kA) \\ &= RS \end{aligned}$$

c) Prove that $A - A^T$ is skew symmetric.

To prove: $(A - A^T)^T = -(A - A^T)$

$$\begin{aligned} LS &= (A - A^T)^T \\ &= A^T - (A^T)^T \\ &= A^T - A \\ &= -(A - A^T) \\ &= RS \end{aligned}$$

3. Prove that if A and B are $n \times n$ matrices such that $A^2 = B^2 = (AB)^2 = I$ then $AB = BA$.

Since , since $A^2 = AA = I$ and $B^2 = BB = I$ then A and B are invertible with $A^{-1} = A$ and $B^{-1} = B$.

Since $(AB)^2 = (AB)(AB) = I$, then AB is invertible and $(AB)^{-1} = AB$.

To prove: $AB = BA$

$$\begin{aligned} LS &= AB \\ &= (AB)^{-1} \\ &= B^{-1}A^{-1} \\ &= BA \\ &= RS \end{aligned}$$

4. Let A be a matrix such that $A^3 = I$.

a) Prove that A is invertible.

$$\begin{aligned} A^3 &= I \\ \det(A^3) &= \det(I) \\ [\det(A)]^3 &= 1 \\ \det(A) &= 1 \end{aligned}$$

Thus, since $\det(A) \neq 0$, then A is invertible.

b) To prove: $(I - 2A)(I + 2A + 4A^2) = I$

$$\begin{aligned} \text{LS} &= (I - 2A)(I + 2A + 4A^2) \\ &= I^2 + 2IA + 4IA^2 - 2AI - 4A^2 - 8A^3 \\ &= I + 2A + 4A^2 - 2A - 4A^2 - 8(0) \\ &= I \\ &= RS \end{aligned}$$

5. Solve the following systems of linear equations, if possible.

a)
$$\begin{aligned} 2x - y + 2z &= 1 \\ x + y - 3z &= 4 \\ 5x - y + z &= 6 \\ x + 4y - 11z &= 11 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -3 & 4 \\ 5 & -1 & 1 & 6 \\ 1 & 4 & -11 & 11 \end{array}\right] \xrightarrow[R_4 \to 2R_4 - R_1]{\substack{R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 - 5R_1}} \left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -8 & 7 \\ 0 & 3 & -8 & 7 \\ 0 & 9 & -24 & 21 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 3R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -8 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{3} R_2]{R_1 \to \frac{1}{2} R_3} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & \frac{-8}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} c_3 &= t \\ c_2 &= \tfrac{7}{3} + \tfrac{8}{3} t \\ c_1 &= \tfrac{5}{3} + \tfrac{1}{3} t \end{aligned}$$

Solution: $\left(\frac{5}{3} + \frac{1}{3} t, \frac{7}{3} + \frac{8}{3} t, t\right)$

b)
$$\begin{aligned} 4x - y + 2z &= 3 \\ 2x - 5y + z &= 9 \\ 2x + 4y + z &= -6 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 2 & 3 \\ 2 & -5 & 1 & 9 \\ 2 & 4 & 1 & -6 \end{array}\right] \xrightarrow[R_3 \to 2R_3 - R_1]{R_2 \to 2R_2 - R_1} \left[\begin{array}{ccc|c} 4 & -1 & 2 & 3 \\ 0 & -9 & 0 & 15 \\ 0 & 9 & 0 & -15 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 4 & -1 & 2 & 3 \\ 0 & -9 & 0 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{-1}{9} R_2]{R_1 \to \frac{1}{4} R_1} \left[\begin{array}{ccc|c} 1 & \frac{-1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 1 & 0 & \frac{-5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} c_3 &= t \\ c_2 &= \tfrac{-5}{3} \\ c_1 &= \tfrac{1}{3} - \tfrac{1}{2} t \end{aligned}$$

Solution: $\left(\frac{1}{3} - \frac{1}{2} t, -\frac{5}{3}, t\right)$

c)
$$\begin{aligned} 2x - y + z - 5t &= 12 \\ 6x + 3y - 2z + 3w + t &= 1 \\ 2x + 5y - 4z + 3w + 11t &= 8 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 0 & -5 & 12 \\ 6 & 3 & -2 & 3 & 1 & 1 \\ 2 & 5 & -4 & 3 & 11 & 8 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccccc|c} 2 & -1 & 1 & 0 & -5 & 12 \\ 0 & 6 & -5 & 3 & 16 & -35 \\ 0 & 6 & -5 & 3 & 16 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccccc|c} 2 & -1 & 1 & 0 & -5 & 12 \\ 0 & 6 & -5 & 3 & 16 & -35 \\ 0 & 0 & 0 & 0 & 0 & 31 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{31} R_3]{\substack{R_1 \to \frac{1}{2} R_1 \\ R_2 \to \frac{1}{6} R_2}} \left[\begin{array}{ccccc|c} 1 & \frac{-1}{2} & \frac{1}{2} & 0 & \frac{-5}{2} & 6 \\ 0 & 1 & \frac{-5}{6} & \frac{1}{2} & \frac{8}{3} & \frac{-35}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

No solutions

d) $3x - y + 2z = 9$
$5x - y + 3z = 16$
$2x + 3y - z = 11$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 9 \\ 5 & -1 & 3 & 16 \\ 2 & 3 & -1 & 11 \end{array}\right] \xrightarrow[R_3 \to 3R_3 - 2R_1]{R_2 \to 3R_2 - 5R_1} \left[\begin{array}{ccc|c} 3 & -1 & 2 & 9 \\ 0 & 2 & -1 & 3 \\ 0 & 11 & -7 & 15 \end{array}\right]$$

$$\xrightarrow{R_3 \to 2R_3 - 11R_2} \left[\begin{array}{ccc|c} 3 & -1 & 2 & 9 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & -3 & -3 \end{array}\right] \xrightarrow[R_3 \to \frac{-1}{3}R_3]{R_1 \to \frac{1}{3}R_1,\ R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & \frac{-1}{3} & \frac{2}{3} & 3 \\ 0 & 1 & \frac{-1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution: $(3,2,1)$

6. For which values of a will the following system of linear equations have

$$\begin{aligned} x - y + 2z &= 7 \\ -2x + ay + -4z &= 5a - 24 \\ 3x + 2y + (6-a)z &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ -2 & a & -4 & 5a-24 \\ 3 & 2 & 6-a & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & a-2 & 0 & 5a-10 \\ 0 & 5 & -a & -17 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 5 & -a & -17 \\ 0 & a-2 & 0 & 5a-10 \end{array}\right] \xrightarrow{R_3 \to 5R_3 - (a-2)R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 5 & -a & -17 \\ 0 & 0 & -a(a-2) & 42a-84 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{-a(a-2)}R_3]{R_2 \to \frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 1 & \frac{-1}{5}a & \frac{-17}{5} \\ 0 & 0 & 1 & \frac{42a-84}{-a(a-2)} \end{array}\right]$$

Illegal if $a(a-2) = 0$
$a = 0$ or 2

a) Thus there will be a unique solution if $a \neq 0, 2$

b) If $a = 0$, $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 5 & 0 & -17 \\ 0 & 0 & 0 & -84 \end{array}\right]$ so there is no solution

c) If $a = 2$, $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 5 & -2 & -17 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{aligned} z &= t \\ y &= \tfrac{-17}{5} \\ x &= \tfrac{52}{5} - 2t \end{aligned}$ an infinite number of solutions

7. Solve the following systems of linear equations using (i) Cramer's rule (ii) the inverse.

$$\text{a)}\quad \begin{aligned} 3x - \ \ y + \quad &= \ 1 \\ 5x \qquad + 3z &= 14 \\ x + 3y - \ \ z &= \ 4 \end{aligned}$$

i – Cramer's Rule

$$\det(A) = \begin{vmatrix} 3 & -1 & 0 \\ 5 & 0 & 3 \\ 1 & 3 & -1 \end{vmatrix} = 3\begin{vmatrix} 0 & 3 \\ 3 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 3 \\ 1 & -1 \end{vmatrix} 0 = -27 - 8 = -35$$

$$\det(A(1)) = \begin{vmatrix} 1 & -1 & 0 \\ 14 & 0 & 3 \\ 4 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 3 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 14 & 3 \\ 4 & -1 \end{vmatrix} + 0 = -9 - 26 = -35$$

$$\det(A(2)) = \begin{vmatrix} 3 & 1 & 0 \\ 5 & 14 & 3 \\ 1 & 4 & -1 \end{vmatrix} = 3\begin{vmatrix} 14 & 3 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 5 & 3 \\ 1 & -1 \end{vmatrix} + 0 = -78 + 8 = -70$$

$$\det(A) = \begin{vmatrix} 3 & -1 & 1 \\ 5 & 0 & 14 \\ 1 & 3 & 4 \end{vmatrix} = 3\begin{vmatrix} 0 & 14 \\ 3 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 14 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 1 & 3 \end{vmatrix} = -126 + 6 + 15 = -105$$

$$x = \frac{\det(A(1))}{\det(A)} = \frac{-35}{-35} = 1 \qquad\qquad y = \frac{\det(A(2))}{\det(A)} = \frac{-70}{-35} = 2$$

$$z = \frac{\det(A(3))}{\det(A)} = \frac{-105}{-35} = 3 \qquad\qquad \text{Solution: } (1,2,3)$$

ii - The inverse.

$$\operatorname{cof}(A) = \begin{bmatrix} -9 & 8 & 15 \\ -1 & -3 & -10 \\ -3 & -9 & 5 \end{bmatrix} \qquad\qquad \operatorname{adj}(A) = \begin{bmatrix} -9 & --1 & -3 \\ 8 & -3 & -9 \\ 15 & -10 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \begin{bmatrix} \frac{9}{35} & \frac{1}{35} & \frac{3}{35} \\ \frac{-8}{35} & \frac{3}{35} & \frac{9}{35} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \end{bmatrix}$$

$$X = A^{-1}b = \begin{bmatrix} \frac{9}{35} & \frac{1}{35} & \frac{3}{35} \\ \frac{-8}{35} & \frac{3}{35} & \frac{9}{35} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \end{bmatrix}\begin{bmatrix} 1 \\ 14 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

b) $$\begin{aligned} 2x + y + 4z &= 8 \\ 2x - y + \quad &= -16 \\ 3x \quad + 5z &= 2 \end{aligned}$$

i – Cramer's Rule

$$\det(A) = \begin{vmatrix} 2 & 1 & 4 \\ 2 & -1 & 0 \\ 3 & 0 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} - 0 + \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = 12 - 4 = 8$$

$$\det(A(1)) = \begin{vmatrix} 8 & 1 & 4 \\ -16 & -1 & 0 \\ 2 & 0 & 5 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} - 0 + \begin{vmatrix} 8 & 1 \\ -16 & -1 \end{vmatrix} = 8 + 8 = 16$$

$$\det(A(2)) = \begin{vmatrix} 2 & 8 & 4 \\ 2 & -16 & 0 \\ 3 & 2 & 5 \end{vmatrix} = -2\begin{vmatrix} 8 & 4 \\ 2 & 5 \end{vmatrix} - 16\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 0 = -64 + 32 = -32$$

$$\det(A(3)) = \begin{vmatrix} 2 & 1 & 8 \\ 2 & -1 & -16 \\ 3 & 0 & 2 \end{vmatrix} = 3\begin{vmatrix} 1 & 8 \\ -1 & -16 \end{vmatrix} - 0 + 2\begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = -24 - 8 = -32$$

$$x = \frac{\det(A(1))}{\det(A)} = \frac{16}{8} = 2 \qquad y = \frac{\det(A(2))}{\det(A)} = \frac{-32}{8} = -4$$

$$z = \frac{\det(A(3))}{\det(A)} = \frac{-32}{8} = -4 \qquad \text{Solution: } (-6, 4, 4)$$

ii - The inverse.

$$\operatorname{cof}(A) = \begin{bmatrix} -5 & -10 & 3 \\ -5 & -2 & 3 \\ 4 & 8 & -4 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} -5 & -5 & 4 \\ -10 & -2 & 8 \\ 3 & 3 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \begin{bmatrix} \frac{5}{8} & \frac{5}{8} & \frac{-1}{2} \\ \frac{5}{4} & \frac{1}{4} & -1 \\ \frac{-3}{8} & \frac{-3}{8} & \frac{1}{2} \end{bmatrix}$$

$$X = A^{-1}b = \begin{bmatrix} \frac{5}{8} & \frac{5}{8} & \frac{-1}{2} \\ \frac{5}{4} & \frac{1}{4} & -1 \\ \frac{-3}{8} & \frac{-3}{8} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 8 \\ -16 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ 4 \end{bmatrix}$$

8. Consider the matrix $A=\begin{bmatrix}3 & -1\\ 5 & 4\end{bmatrix}$.

a) Find the inverse of A using the adjoint.

$$\operatorname{cof}(A)=\begin{bmatrix}4 & -5\\ 1 & 3\end{bmatrix} \qquad \operatorname{adj}(A)=\begin{bmatrix}4 & 1\\ -5 & 3\end{bmatrix} \qquad \det(A)=17$$

$$A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\begin{bmatrix}\frac{4}{17} & \frac{1}{17}\\ \frac{-5}{17} & \frac{3}{17}\end{bmatrix}$$

b) Express A^{-1} as a product of elementary matrices.

$$\left[\begin{array}{cc|cc}3 & -1 & 1 & 0\\ 5 & 4 & 0 & 1\end{array}\right]\xrightarrow{R_1\to\frac{1}{3}R_1}\left[\begin{array}{cc|cc}1 & \frac{-1}{3} & \frac{1}{3} & 0\\ 5 & 4 & 0 & 1\end{array}\right] \qquad E_1=\begin{bmatrix}\frac{1}{3} & 0\\ 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-5R_1}\left[\begin{array}{cc|cc}1 & \frac{-1}{3} & \frac{1}{3} & 0\\ 0 & \frac{17}{3} & \frac{-5}{3} & 1\end{array}\right] \qquad E_2=\begin{bmatrix}1 & 0\\ -5 & 1\end{bmatrix}$$

$$\xrightarrow{R_2\to\frac{3}{17}R_2}\left[\begin{array}{cc|cc}1 & \frac{-1}{3} & \frac{1}{3} & 0\\ 0 & 1 & \frac{-5}{17} & \frac{3}{17}\end{array}\right] \qquad E_3=\begin{bmatrix}1 & 0\\ 0 & \frac{3}{17}\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1+\frac{1}{3}R_2}\left[\begin{array}{cc|cc}1 & 0 & \frac{4}{17} & \frac{1}{17}\\ 0 & 1 & \frac{-5}{17} & \frac{3}{17}\end{array}\right] \qquad E_4=\begin{bmatrix}1 & \frac{1}{3}\\ 0 & 1\end{bmatrix}$$

$$A^{-1}=E_4E_3E_2E_1$$

$$\begin{bmatrix}\frac{4}{17} & \frac{1}{17}\\ \frac{-5}{17} & \frac{3}{17}\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{3}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \frac{3}{17}\end{bmatrix}\begin{bmatrix}1 & 0\\ -5 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{3} & 0\\ 0 & 1\end{bmatrix}$$

c) Express A as a product of elementary matrices.

$$A=E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

$$\begin{bmatrix}3 & -1\\ 5 & 4\end{bmatrix}=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \frac{17}{3}\end{bmatrix}\begin{bmatrix}1 & \frac{-1}{3}\\ 0 & 1\end{bmatrix}$$

9. Consider the matrix $A=\begin{bmatrix}2 & 0 & 1\\ 4 & 1 & 2\\ 3 & 3 & 0\end{bmatrix}$.

a) Find the inverse of A using the adjoint.

$$\operatorname{cof}(A)=\begin{bmatrix}-6 & 6 & 9\\ 3 & -3 & -6\\ -1 & 0 & 2\end{bmatrix} \qquad \operatorname{adj}(A)=\begin{bmatrix}-6 & 3 & -1\\ 6 & -3 & 0\\ 9 & -6 & 2\end{bmatrix}$$

$$\det(A)=\begin{vmatrix}2 & 0 & 1\\ 4 & 1 & 2\\ 3 & 3 & 0\end{vmatrix}=2\begin{vmatrix}1 & 2\\ 3 & 0\end{vmatrix}-0+\begin{vmatrix}4 & 1\\ 3 & 3\end{vmatrix}=-12+9=-3$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \begin{bmatrix} 2 & -1 & \frac{1}{3} \\ -2 & 1 & 0 \\ -3 & 2 & \frac{-2}{3} \end{bmatrix}$$

b) Express A^{-1} and A as a product of elementary matrices.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \qquad E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 3 & \frac{-3}{2} & \frac{-3}{2} & 0 & 1 \end{array}\right] \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & \frac{-3}{2} & \frac{9}{2} & -3 & 1 \end{array}\right] \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{-2}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 2 & \frac{-2}{3} \end{array}\right] \qquad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-2}{3} \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 2 & -1 & \frac{1}{3} \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 2 & \frac{-2}{3} \end{array}\right] \qquad E_6 = \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = E_6 E_5 E_4 E_3 E_2 E_1$$

$$\begin{bmatrix} 2 & -1 & \frac{1}{3} \\ -2 & 1 & 0 \\ -3 & 2 & \frac{-2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-3}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. A coin bank has only nickels, dimes and quarters. The value of the coins is \$2. There are twice as many nickles as dimes and one more dime than quarters. Find the number of each coin in the bank.

$$5x+10y+25z=200$$
$$x-2y=0$$
$$y-z=1$$

$$\left[\begin{array}{ccc|c} 5 & 10 & 25 & 200 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_2 \to 5R_2 - R_1} \left[\begin{array}{ccc|c} 5 & 10 & 25 & 200 \\ 0 & -20 & -25 & -200 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to 20R_3 + R_2} \left[\begin{array}{ccc|c} 5 & 10 & 25 & 200 \\ 0 & -20 & -25 & -200 \\ 0 & 0 & -45 & -180 \end{array}\right] \xrightarrow[R_3 \to \frac{-1}{45}R_3]{R_1 \to \frac{1}{5}R_1,\ R_2 \to \frac{-1}{20}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 5 & 40 \\ 0 & 1 & \frac{5}{4} & 10 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$z=4,\ y=5,\ x=10$

Thus there are 10 nickels, 5 dimes and 4 quarters in the bank.

11. Suppose a man has three modes of transportation to work: he can walk, drive his car, or take the bus. If he walks one day, then he will either take the car the next day with a probability of $\frac{2}{3}$ or take the bus with a probability of $\frac{1}{3}$. If he drove one day, then he will walk the next day with a probability of $\frac{1}{2}$ and take the bus with a probability of $\frac{1}{2}$. If he took the bus one day, then he will walk the next day with a probability of $\frac{1}{3}$, drive with a probability of $\frac{1}{3}$ and take the bus with a probability of $\frac{1}{3}$.

a) Find the transition matrix. $A=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

b) If he drives on Monday, find the probability that he will walk, drive or take the bus on Wednesday.

$$X_1=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \qquad X_2=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}=\begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

Walk: $\frac{1}{6}$ Drive: $\frac{1}{2}$ Bus: $\frac{1}{3}$

c) In the long run, how often will he walk, drive and take the bus?

$$(I-A)X=0$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{3} & 0 \\ \frac{-2}{3} & 1 & \frac{-1}{3} & 0 \\ \frac{-1}{3} & \frac{-1}{2} & \frac{2}{3} & 0 \end{array}\right] \xrightarrow[R_3 \to 3R_3 + R_1]{R_2 \to 3R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{3} & 0 \\ 0 & 2 & \frac{-5}{3} & 0 \\ 0 & -2 & \frac{5}{3} & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{3} & 0 \\ 0 & 2 & \frac{-5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2}\left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{3} & 0 \\ 0 & 1 & \frac{-5}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} z &= t \\ y &= \tfrac{5}{6}t \\ x &= \tfrac{3}{4}t \end{aligned}$$

$$\begin{aligned} \tfrac{3}{4}t + \tfrac{5}{6}t + t &= 1 \\ t &= \tfrac{12}{31} \end{aligned}$$

Thus $X = \left(\frac{9}{31}, \frac{10}{31}, \frac{12}{31}\right)$

Walk: $\frac{9}{31}$ Drive: $\frac{10}{31}$ Bus: $\frac{12}{31}$

12. Find the equation of the parabola passing through the points $A(1,4)$, $B(2,12)$ and $C(-3,32)$.

$y = a + bx + cx^2$

$$\begin{aligned} 4 &= 1 + x + x^2 \\ 12 &= 1 + 2x + 4x^2 \\ 32 &= 1 - 3x + 9x^2 \end{aligned}$$

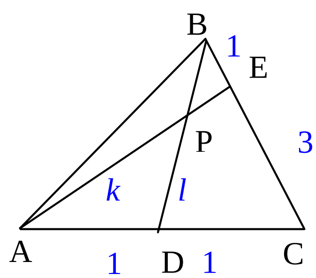
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 12 \\ 1 & -3 & 9 & 32 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 8 \\ 0 & -4 & 8 & 28 \end{array}\right] \xrightarrow{R_3 \to R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 20 & 60 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{20}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{aligned} c &= 3 \\ b &= -1 \\ a &= 2 \end{aligned}$$

Thus the parabola is $y = 2 - x + 3x^2$.

13. Let ABC be a triangle and E a point on the segment BC dividing it in a ration of 1 to 3. Let D be the midpoint of AC. Join A to E and B to D, and let P be the point on intersection of the segments AE and BD. In what ration does P divide AE and BD?

Let $\overrightarrow{AP} = k\overrightarrow{AG}$ and $\overrightarrow{FP} = l\overrightarrow{FE}$.



We have

$$\begin{aligned} \overrightarrow{BE} &= \tfrac{1}{4}\overrightarrow{BC} \\ \overrightarrow{AD} &= \tfrac{1}{2}\overrightarrow{AC} \end{aligned}$$

Let us express $\overrightarrow{AP}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in two different ways.

$$\begin{aligned}\overrightarrow{AP} &= k\overrightarrow{AE} \\ &= k\left(\overrightarrow{AB}+\overrightarrow{BE}\right) \\ &= k\overrightarrow{AB}+\tfrac{1}{4}k\overrightarrow{BC} \\ &= k\overrightarrow{AB}+\tfrac{k}{4}\left(\overrightarrow{BA}+\overrightarrow{AC}\right) \\ &= \tfrac{3k}{4}\overrightarrow{AB}+\tfrac{k}{4}\overrightarrow{AC}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{AD}+\overrightarrow{DP} \\ &= \tfrac{1}{2}\overrightarrow{AC}+l\overrightarrow{DB} \\ &= \tfrac{1}{2}\overrightarrow{AC}+l\left(\overrightarrow{DA}+\overrightarrow{AB}\right) \\ &= \tfrac{1}{2}\overrightarrow{AC}+l\left(\tfrac{-1}{2}\overrightarrow{AC}+\overrightarrow{AB}\right) \\ &= \left(\tfrac{1}{2}-\tfrac{1}{2}l\right)\overrightarrow{AC}+l\overrightarrow{AB}\end{aligned}$$

By the basis theorem, we have the equations

$$\tfrac{3k}{4} = l \qquad \tfrac{1}{4}k = \tfrac{1}{2}-\tfrac{1}{2}l$$

Combining these equations, we have

$$\begin{aligned}\tfrac{1}{4}k &= \tfrac{1}{2}-\tfrac{1}{2}\tfrac{3k}{4} \\ \tfrac{5}{8}k &= \tfrac{1}{2} \\ k &= \tfrac{4}{5}\end{aligned}$$

thus $k=\frac{4}{5}$ and $l=\frac{3}{5}$.

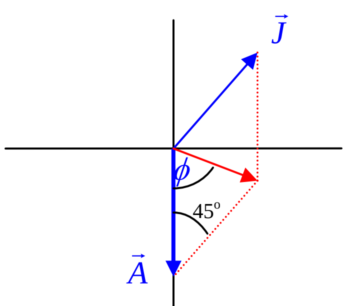
Hence, P divides AE in a ration of 4 to 1 and divides DB in a ration of 3 to 2.

14. Let ABC be a triangle and M, N and P the midpoints of AB, BC and CA respectively. Prove that if O is any point (inside or outside the triangle) then

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}$$

$$\begin{aligned}\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC} &= \left(\overrightarrow{OM}+\overrightarrow{MA}\right)+\left(\overrightarrow{ON}+\overrightarrow{NB}\right)+\left(\overrightarrow{OP}+\overrightarrow{PC}\right) \\ &= \overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}+\overrightarrow{MA}+\overrightarrow{NB}+\overrightarrow{PC} \\ &= \overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}+\tfrac{1}{2}\overrightarrow{BA}+\tfrac{1}{2}\overrightarrow{CB}+\tfrac{1}{2}\overrightarrow{AC} \\ &= \overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}+\tfrac{1}{2}\left(\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BA}\right) \\ &= \overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}+\tfrac{1}{2}\vec{0} \\ &= \overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}\end{aligned}$$

15. A Boeing 737 aircraft maintains a constant airspeed of 500 miles per hour in the direction due south. The velocity of the jet stream is 80 miles per hour in a northeasterly direction (N45^{o}E). Find the actual speed and direction of the aircraft relative to the ground.



$$\begin{aligned}\left\|\vec{A}+\vec{J}\right\|^2 &= \left\|\vec{A}\right\|^2+\left\|\vec{J}\right\|^2+2\left\|\vec{A}\right\|\left\|\vec{J}\right\|\cos\theta \\ &= 500^2+80^2-2\cdot 500\cdot 80\cos 45° \\ &= 199831\end{aligned}$$

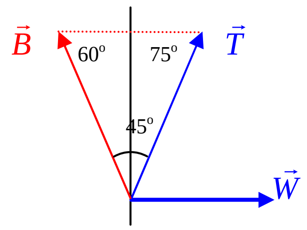
$$\left\|\vec{A}+\vec{J}\right\| = 447$$

$$\frac{\sin 45}{447}=\frac{\sin\phi}{80} \qquad \phi = 7.3°$$

Thus the plane has a speed of 447 miles per hour in a S7.3^{o}E direction.

16. A river flows from west to east. There are ferry terminals on the north and south shores, the north dock being 15° east of north (i.e.N15°E) from the south dock. The ferry captain knows from experience that in order to reach the dock on the north shore from the south shore dock, she has to steer N30°W.

a) If the ferry travels at 12 km/h, what is the speed of the current?



$$\frac{\sin 45°}{\|\vec{W}\|} = \frac{\sin 75°}{12}$$

$$\|\vec{W}\| = 8.78$$

Speed of current is 8.78 km/h

b) If the trip takes ¼ hour, how far apart are the dock?

$$\frac{\sin 60°}{\|\vec{T}\|} = \frac{\sin 75°}{12} \qquad \|\vec{T}\| = 10.76 \text{ km/h}$$

Thus the distance between the docks is $\frac{1}{4}10.76 = 2.69$ km

17. Consider the vectors $\vec{u} = (-2,5,5)$, $\vec{v} = (1,-1,2)$ and $\vec{w} = (5,-1,2)$.

a) Evaluate $2\vec{u} - 3\vec{v}$

$$2\vec{u} - 3\vec{v} = 2(-2,5,5) - 3(1,-1,2) = (-7,13,4)$$

b) Find the vector projection of $\vec{u}$ onto $\vec{w}$.

$$\text{proj}_{\vec{w}}\,\vec{u} = \frac{\vec{u}\bullet\vec{w}}{\vec{w}\bullet\vec{w}}\vec{w} = \frac{-5}{30}(5,-1,2) = \left(\frac{-5}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

c) Find the angle between $\vec{u}$ and $\vec{w}$.

$$\cos\theta = \frac{\vec{u}\bullet\vec{w}}{\|\vec{u}\|\|\vec{w}\|} = \frac{-5}{\sqrt{54}\sqrt{30}} \qquad \theta \approx 97.1°$$

d) Find the area of the triangle having sides $\vec{u}$ and $\vec{v}$.

$$A = \tfrac{1}{2}\|\vec{u}\times\vec{v}\| = \tfrac{1}{2}\begin{vmatrix} i & j & k \\ -2 & 5 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \tfrac{1}{2}\|(15,9,-3)\| = \frac{3\sqrt{35}}{2}$$

e) Find the volume of the parallelepiped having sides $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$$V = |\vec{u}\bullet(\vec{v}\times\vec{w})| = \left\|\begin{matrix} -2 & 5 & 5 \\ 1 & -1 & 2 \\ 5 & -1 & 2 \end{matrix}\right\| = \left|-2\begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} - 5\begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} + 5\begin{vmatrix} 1 & -1 \\ 5 & -1 \end{vmatrix}\right| = |0+40+20| = 60$$

18. Consider the points A(2,-1,3), and B(3,-1,5), C(-2,2,3) and D(-1,0,5).

a) Find the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.

$$\text{proj}_{\overrightarrow{AC}}\,\overrightarrow{AB} = \frac{\overrightarrow{AB}\bullet\overrightarrow{AC}}{\overrightarrow{AC}\bullet\overrightarrow{AC}}\overrightarrow{AC} = \frac{(1,0,2)\bullet(-4,3,0)}{(-4,3,0)\bullet(-4,3,0)}(-4,3,0) = \frac{-4}{25}(-4,3,0) = \left(\frac{16}{25}, \frac{-12}{25}, 0\right)$$

b) Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\cos\theta = \frac{\overrightarrow{AB}\bullet\overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|} = \frac{-4}{\sqrt{5}\sqrt{25}} \qquad \theta \approx 111$$

c) Find the volume of the tetrahedron ABCD.

$$V = \tfrac{1}{6}\left|\overrightarrow{AB}\bullet\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right| = \tfrac{1}{6}\left\|\begin{matrix} 1 & 0 & 2 \\ -4 & 3 & 0 \\ -3 & 1 & 2 \end{matrix}\right\| = \tfrac{1}{6}\left|1\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} - 0 + 2\begin{vmatrix} -4 & 3 \\ -3 & 1 \end{vmatrix}\right| = \tfrac{1}{6}\left|6+10\right| = \frac{8}{3}$$

d) Find the equation of the line (in ***parametric form***) passing through *D* and parallel to $\overrightarrow{AB}$.

$$\vec{u} = \overrightarrow{AB} = (1,0,2) \qquad l : \begin{cases} x = -1+t \\ y = 0 \\ z = 5+2t \end{cases}$$

e) Find the equation of the plane (in ***general form***) parallel to $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and passing through *D*.

$$\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ -4 & 3 & 0 \end{vmatrix} = (-6,-8,3)$$

$$-6x-8x+3z = -6(-1)-8(0)+3(5) = 21$$

$$\pi : 6x+8y-3z = -21$$

f) Find the equation of the plane perpendicular to $\overrightarrow{AC}$ and passing through *D*.

$$\vec{n} = \overrightarrow{AC} = (-4,3,0) \qquad -4x+3y = -4(-1)+3(0)+0(5) = 4$$

$$\pi : 4x-3y = -4$$

19. Consider the plane $\pi : 2x+y-5z+1=0$ and the line $L : \dfrac{x-1}{3} = \dfrac{2y+1}{4} = 3-z$

a) Find the equation of the line (in ***symmetric form***) perpendicular to π and passing through P(1,1,-3).

$$\vec{u} = (2,1,-5) \qquad \frac{x-1}{2} = y-1 = \frac{z+3}{-5}$$

b) Find the equation of the line (in ***parametric form***) parallel to *L* and passing through P(1,1,-3).

$$\vec{u} = (3,2,-1) \qquad \begin{cases} x = 1+3t \\ y = 1+2t \\ z = -3-t \end{cases}$$

c) Find the distance between the line L and the line found in (b).

Since the lines are parallel, we have

$P_1\left(1,\frac{-1}{2},3\right)$ $\qquad \overrightarrow{P_1P_2}=\left(0,\frac{3}{2},-6\right)$

$P_2\left(1,1,-3\right)$

$$d=\frac{\left\|\overrightarrow{P_1P_2}\times\vec{u}\right\|}{\left\|\vec{u}\right\|}=\frac{\left\|\begin{vmatrix} i & j & k \\ 0 & \frac{3}{2} & -6 \\ 3 & 2 & -1\end{vmatrix}\right\|}{\sqrt{14}}=\frac{\left\|\left(\frac{21}{2},-18,\frac{-9}{2}\right)\right\|}{\sqrt{14}}=\frac{\frac{3}{2}\sqrt{202}}{\sqrt{14}}=\frac{3}{14}\sqrt{707}$$

d) Find the intersection, if possible, of the plane π and the line L.

$$L:=\begin{cases} x=1+3t \\ y=\frac{-1}{2}+2t \\ z=3-t\end{cases}$$

$$2\left(1+3t\right)+\left(\frac{-1}{2}+2t\right)-5\left(3-t\right)+1=0$$
$$13t=\frac{27}{2}$$
$$t=\frac{27}{26}$$

Intersection: $\left(1+2\frac{27}{26},\frac{-1}{2}+2\frac{27}{26},3-\frac{27}{26}\right)=\left(\frac{101}{26},\frac{37}{26},\frac{53}{26}\right)$

e) Find the equation of the plane π in vector form.

$z=t$

$y=s$

$x=-\frac{1}{2}-\frac{1}{2}s+\frac{5}{2}t$

$$\pi:\left(x,y,z\right)=\left(\frac{-1}{2},0,0\right)+s\left(\frac{-1}{2},1,0\right)+t\left(\frac{5}{2},0,1\right)$$

f) Find the equation of the plane π_2 (in ***general form***) perpendicular to L and passing through P(1,1,-3).

$\vec{n}_2=\left(3,2,-1\right)$ $\qquad 3x+2y-z=3\left(1\right)+2\left(1\right)-\left(-3\right)=8$

$\pi_2:3x+2y-z=8$

g) Find the angle between the plane π and the plane π_2 found in (f).

$$\cos\theta=\frac{\left|\vec{n}_1\bullet\vec{n}_2\right|}{\left\|\vec{n}_1\right\|\left\|\vec{n}_2\right\|}=\frac{\left|13\right|}{\sqrt{30}\sqrt{14}} \qquad \theta\approx 50.6^\circ$$

h) Find the point Q on the plane π that is closest to the point P(1,1,-3).

$R\left(0,-1,0\right)$ $\qquad \overrightarrow{PQ}=\text{proj}_{\vec{n}}\overrightarrow{PR}=\frac{\overrightarrow{PR}\bullet\vec{n}}{\vec{n}\bullet\vec{n}}\vec{n}$

$\overrightarrow{PR}=\left(-1,-2,3\right)$ $\qquad \left(x-1,y-1,z+3\right)=\frac{-19}{30}\left(2,1,-5\right)$

$\vec{n}=\left(2,1,-5\right)$

$$\left(x,y,z\right)=\frac{-19}{30}\left(2,1,-5\right)+\left(1,1,-3\right)=\left(\frac{-4}{15},\frac{11}{30},\frac{1}{6}\right)$$

$Q\left(\frac{-4}{15},\frac{11}{30},\frac{1}{6}\right)$

i) Find the point Q on the line L that is closest to the point P(1,1,-3).

$$R\left(1,\tfrac{-1}{2},3\right) \qquad \overrightarrow{RQ}=\text{proj}_{\vec{u}}\,\overrightarrow{RP}=\frac{\overrightarrow{RP}\bullet\vec{u}}{\vec{u}\bullet\vec{u}}\vec{u}$$

$$\overrightarrow{RP}=\left(0,\tfrac{3}{2},-6\right)$$

$$\vec{u}=\left(3,2,-1\right) \qquad \left(x-1,y-1,z+3\right)=\frac{9}{14}\left(3,2,-1\right)$$

$$\left(x,y,z\right)=\frac{9}{14}\left(3,2,-1\right)+\left(1,1,-3\right)=\left(\tfrac{41}{14},\tfrac{11}{14},\tfrac{33}{14}\right)$$

$$Q\left(\tfrac{41}{14},\tfrac{11}{14},\tfrac{33}{14}\right)$$

j) Find the distance from the point P(1,1,-3) to the plane π.

$$R\left(0,-1,0\right)$$

$$\overrightarrow{PR}=\left(-1,-2,3\right) \qquad d=\frac{\left|\overrightarrow{PR}\bullet\vec{n}\right|}{\left\|\vec{n}\right\|}=\frac{\left|-19\right|}{\sqrt{30}}=\frac{19\sqrt{30}}{30}$$

k) Find the distance from the point P(1,1,-3) to the line L.

$$R\left(1,\tfrac{-1}{2},3\right) \qquad \overrightarrow{PR}\times\vec{u}=\begin{vmatrix} i & j & k \\ 0 & \tfrac{-3}{2} & 6 \\ 3 & 2 & -1 \end{vmatrix}=\left(\tfrac{-21}{2},18,\tfrac{9}{2}\right)$$

$$\overrightarrow{PR}=\left(0,\tfrac{-3}{2},6\right)$$

$$d=\frac{\left\|\overrightarrow{PR}\times\vec{u}\right\|}{\left\|\vec{u}\right\|}=\frac{\left\|\left(\tfrac{-21}{2},18,\tfrac{9}{2}\right)\right\|}{\sqrt{14}}=\frac{\tfrac{3}{2}\sqrt{202}}{\sqrt{14}}=\frac{3\sqrt{707}}{14}$$

l) Find the equation of the plane (if possible), in general form containing the lines L and $L_2:\dfrac{x+5}{3}=\dfrac{y-2}{2}=-z$.

$$\vec{u}=\left(3,2,-1\right)$$

$$\vec{u}_2=\left(3,2,-1\right) \qquad \therefore L \parallel L_2$$

$P\left(1,\tfrac{-1}{2},3\right)\in L$, Since $\dfrac{1+5}{3}\neq\dfrac{\tfrac{-1}{2}-2}{2}\neq -3$ then $P\notin L_2$

Thus L and L_2 are parallel and distinct.

$$P_2\left(-5,2,0\right) \qquad \vec{n}=\vec{u}\times\overrightarrow{PP_2}=\begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -6 & \tfrac{5}{2} & -3 \end{vmatrix}=\left(\tfrac{-7}{2},15,\tfrac{39}{2}\right)$$

$$\overrightarrow{PP_2}=\left(-6,\tfrac{5}{2},-3\right)$$

$$\tfrac{-7}{2}x+15y-\tfrac{39}{2}z=\tfrac{-7}{2}\left(-5\right)+15\left(2\right)-0=\tfrac{95}{2} \qquad \text{Plane: } 7x-30y-39z=-95$$

m) Find the equation of the plane (if possible), in general form containing the lines L and $L_3:\dfrac{x+4}{2}=y+\tfrac{5}{2}=\dfrac{z+10}{3}$.

$$\vec{u}=\left(3,2,-1\right)$$

$$\vec{u}_3=\left(2,1,3\right) \qquad \therefore L \nparallel L_3$$

To find $L\cap L_3$, we have

$$L:(x,y,z)=\left(1,\tfrac{-1}{2},3\right)+t(3,2,-1)$$
$$L_3:(x,y,z)=\left(-4,\tfrac{-5}{2},-10\right)+s(2,1,3)$$
$$\left(1,\tfrac{-1}{2},3\right)+t(3,2,-1)=\left(-4,\tfrac{-5}{2},-10\right)+s(2,1,3)$$
$$t(3,2,-1)+s(-2,-1,-3)=(-5,-2,-13)$$

$$\left[\begin{array}{cc|c}3&-2&-5\\2&-1&-2\\-1&-3&-13\end{array}\right]\xrightarrow[R_3\to 3R_3+R_1]{R_2\to 3R_2-2R_1}\left[\begin{array}{cc|c}3&-2&-5\\0&1&4\\0&-11&-44\end{array}\right]\xrightarrow{R_3\to 11R_3+R_2}\left[\begin{array}{cc|c}3&-2&-5\\0&1&4\\0&0&0\end{array}\right]$$

$$\xrightarrow{R_1\to\frac{1}{3}R_1}\left[\begin{array}{cc|c}1&\frac{-2}{3}&\frac{-5}{3}\\0&1&4\\0&0&0\end{array}\right]\qquad \begin{array}{l}s=4\\t=1\end{array}$$

$$(x,y,z)=\left(1,\tfrac{-1}{2},3\right)+(3,2,-1)=\left(4,\tfrac{3}{2},2\right)$$

Thus $L\cap L_3=P\left(4,\tfrac{3}{2},2\right)$, so the lines are nonparallel and intersecting.

$$\vec{n}=\vec{u}\times\vec{u}_3=\begin{vmatrix}i&j&k\\3&2&-1\\2&1&3\end{vmatrix}=(7,-11,-1)\qquad 7x-11y-z=7(4)-11\left(\tfrac{3}{2}\right)-2=\tfrac{19}{2}$$

Plane: $7x-11y-z=\frac{19}{2}$

20. Is the set V a vector space with the following operations?

a) $V=\mathbb{R}^2$ $\quad (u_1,u_2)\oplus(v_1,v_2)=(u_1-v_1,u_2-v_2)$

$$k\odot(u_1,u_2)=(ku_1,ku_2)$$

No, Axiom 2 fails: Counter example with $\vec{u}=(1,2)$ and $\vec{v}=(3,4)$

$$(1,2)\oplus(3,4)=(1-3,2-4)=(-2,-2)$$
$$(3,4)\oplus(1,2)=(3-1,4-2)=(2,2)$$

Thus $(1,2)\oplus(3,4)\neq(3,4)\oplus(1,2)$, that is $\vec{u}\oplus\vec{v}\neq\vec{v}\oplus\vec{u}$.

b) $V=\mathbb{R}^2$ $\quad (u_1,u_2)\oplus(v_1,v_2)=(u_1+v_1+1,u_2+v_2+1)$

$$k\odot(u_1,u_2)=(k+ku_1-1,k+ku_2-1)$$

Yes. Let $\vec{u}=(u_1,u_2)$, $\vec{v}=(v_1,v_2)$ and $\vec{w}=(w_1,w_2)$ be in $\mathbb{R}^2$.

1. $\vec{u}\oplus\vec{v}=(u_1+v_1+1,u_2+v_2+1)\in\mathbb{R}^2$
2. $\vec{u}\oplus\vec{v}=(u_1+v_1+1,u_2+v_2+1)=(v_1+u_1+1,v_2+u_2+1)=\vec{v}\oplus\vec{u}$

3. $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (u_1, u_2) \oplus (v_1 + w_1 + 1, v_2 + w_2 + 1)$

$$= (u_1 + (v_1 + w_1 + 1) + 1, u_2 + (v_2 + w_2 + 1) + 1)$$
$$= ((u_1 + v_1 + 1) + w_1 + 1, (u_2 + v_2 + 1) + w_2 + 1)$$
$$= (u_1 + v_1 + 1, u_2 + v_2 + 1) \oplus (w_1, w_2)$$
$$= (\vec{u} \oplus \vec{v}) \oplus \vec{w}$$

4. $\vec{0} = 0 \odot (u_1, u_2) = (-1, -1)$

$\vec{u} \oplus \vec{0} = (u_1, u_2) \oplus (-1, -1) = (u_1 - 1 + 1, u_2 - 1 + 1) = (u_1, u_2) = \vec{u}$

5. $-\vec{u} = (-1) \odot (u_1, u_2) = (-1 - u_1 - 1, -1 - u_2 - 1) = (-2 - u_1, -2 - u_1)$

$\vec{u} \oplus -\vec{u} = (u_1, u_2) \oplus (-2 - u_1, -2 - u_1)$

$$= (u_1 - 2 - u_1 + 1, u_2 - 2 - u_2 + 1) = (-1, -1) = \vec{0}$$

6. $k \odot (u_1, u_2) = (k + ku_1 - 1, k + ku_2 - 1) \in \mathbb{R}^2$

7. $(k + l) \odot \vec{u} = (k + l) \odot (u_1, u_2) = ((k + l) + (k + l)u_1 - 1, (k + 1) + (k + l)u_2 - 1)$

$$= ((k + ku_1 - 1) + (l + lu_1 - 1) + 1, (k + ku_2 - 1) + (l + lu_2 - 1) + 1)$$
$$= (k + ku_1 - 1, k + ku_2 - 1) \oplus (l + lu_1 - 1, l + lu_2 - 1)$$
$$= (k \odot \vec{u}) \oplus (l \odot \vec{u})$$

8. $k \odot (\vec{u} \oplus \vec{v}) = k \odot (u_1 + v_1 + 1, u_2 + v_2 + 1)$

$$= (k + k(u_1 + v_1 + 1) - 1, l + l(u_2 + v_2 + 1) - 1)$$
$$= ((k + ku_1 - 1) + (k + kv_1 - 1) + 1, (k + ku_2 - 1) + (k + kv_2 - 1) + 1)$$
$$= (k + ku_1 - 1, k + ku_2 - 1) \oplus (k + kv_1 - 1, k + kv_2 - 1)$$
$$= (k \odot \vec{u}) \oplus (k \odot \vec{v})$$

9. $(kl) \odot \vec{u} = (kl + klu_1 - 1, kl + klu_2 - 1)$

$$= (k + k(l + lu_1 - 1) - 1, k + k(l + lu_2 - 1) - 1)$$
$$= k \odot (l + lu_1 - 1, l + lu_2 - 1)$$
$$= k \odot (l \odot \vec{u})$$

10. $1 \odot \vec{u} = 1 \odot (u_1, u_2) = (1 + u_1 - 1, 1 + u_2 - 1) = (u_1, u_2) = \vec{u}$

21. Is the set *W* a subspace of *V*? Support your answer.

a) $W = \left\{ \begin{bmatrix} a & a+b \\ a-b & b \end{bmatrix} : a,b \in \mathbb{R} \right\}$ $V = M_{2,2}$

Yes W is nonempty since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$.

Let $A = \begin{bmatrix} a & a+b \\ a-b & b \end{bmatrix}$ and $B = \begin{bmatrix} r & r+s \\ r-s & s \end{bmatrix} \in W$.

1. $A+B = \begin{bmatrix} a+r & a+b+r+s \\ a-b+r-s & b+s \end{bmatrix} = \begin{bmatrix} a+r & (a+r)+(b+s) \\ (a+r)-(b+s) & b+s \end{bmatrix} \in W$

2. $kA = \begin{bmatrix} ka & k(a+b) \\ k(a-b) & kb \end{bmatrix} = \begin{bmatrix} ka & ka+kb \\ ka-kb & kb \end{bmatrix} \in W$

b) $W = \{ A : A \text{ is nilpotent, } A \in M_{2,2} \}$ $V = M_{2,2}$ (A is ***nilpotent*** if $A^2 = 0$)

No. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $A^2 = 0$ and $B^2 = 0$, so $A, B \in V$ but

$$A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin V \text{ since } (A+B)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$$

c) $W = \{ ax^3 - b : a,b \in \mathbb{R} \}$ $V = P_3$

Yes W is nonempty since $0 \in W$.

Let $p(x) = ax^3 - b$ and $q(x) = cx^3 - d \in W$

1. $p(x) + q(x) = ax^3 - b + cx^3 - d = (a+c)x^3 - (b+d) \in W$

2. $kp(x) = k(ax^3 - b) = akx^3 - bk \in W$

d) $W = \{ p(x) : p(1) = p(2),\ p(x) \in P_2 \}$ $V = P_2$

Yes V is nonempty since $p(x) = 0 \in V$ since $p(1) = 0 = p(2)$.

Let $p(x), q(x) \in V$. Then $p(1) = p(2)$ and $q(1) = q(2)$.

1. $(p+q)(x) \in V$ since $(p+q)(1) = p(1) + q(1) = p(2) + q(2) = (p+q)(2)$

2. $(kp)(x) \in V$ since $(kp)(1) = kp(1) = kp(2) = (kp)(2)$

e) $W = \{ (x,y,z) : 2x - 3y + z = 0,\ \ x,y,z \in \mathbb{R} \}$ $V = \mathbb{R}^3$

Yes W is nonempty since $\vec{0} \in W$

Let $\vec{u} = (x_1, y_1, z_1) \in W$ and $\vec{v} = (x_2, y_2, z_2) \in W$.

Then $2x_1 - 3y_1 + z_1 = 0$ and $2x_2 - 3y_2 + z_2 = 0$

1. $\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$

since $2(x_1 + x_2) - 3(y_1 + y_2) + (z_1 + z_2) = (2x_1 - 3y_1 + z_1) + (2x_2 - 3y_2 + z_2)$
$= 0 + 0 = 0$

2. $k\vec{u} = (kx_1, ky_1, kz_1) \in W$ since $2(kx_1) - 3(ky_1) + kz_1 = k(2x_1 - 3y_1 + z_1) = k0 = 0$

f) $W = \{(x,y,z) : 2x - 3y + z - 8 = 0, \quad x,y,z \in \mathbb{R}\}$ $\qquad V = \mathbb{R}^3$

No. If $\vec{u} = (4,0,0) \in W$ and $k = 0$,

then $k\vec{u} = (0,0,0) \notin W$ since $2 \cdot 0 - 3 \cdot 0 + 0 - 8 \neq 0$

Geometrically, W is the plane $2x - 3y + z = 8$.

g) $W = \{(a, 3a-4, a+1) : a \in \mathbb{R}\}$ $\qquad V = \mathbb{R}^3$

No. If $\vec{u} = (1,-1,2) \in W$ and $k = 0$,

then $k\vec{u} = (0,0,0) \notin W$ since if $(0,0,0) = (a, 3a-4, a+1)$, then we have $a = 0$, $a = \frac{4}{3}$ and $a = -1$ which is impossible.

22. For each of the subsets W in $\mathbb{R}^3$,

i) Find a basis for W.
ii) Find the dimension of W
iii) Give a geometrical interpretation of W.

a) $W = \{(2a+b, a, a-2b) : \ a,b \in \mathbb{R}\}$

Since $(2a+b, a, a-2b) = a(2,1,1) + b(1,0,-2)$

Then if $B = \{(2,1,1),(1,0.-2)\}$ we have $W = \text{span}(B)$.

Since B is linearly independent (the two vectors are not multiples of each other) then B is a basis for W.

$\dim(W) = 2$

$$\vec{n} = (2,1,1) \times (1,0,-2) = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix} = (-2,5,-1)$$

Ergo, W is the plane $\pi : 2x - 5y + z = 0$.

b) $W = \{(a-2b+3c, 3a+b+2c, a+4b-3c) : \ a,b,c \in \mathbb{R}\}$

Since $(a-2b+3c, 3a+b+2c, a+4b-3c) = a(1,3,1) + b(-2,1,4) + c(3,2,-3)$

then if $B = \{(1,3,1),(-2,1,4),(3,2,-3)\}$ we have $W = \text{span}(B)$

$$c_1(1,3,1) + c_2(-2,1,4) + c_3(3,2,-3) = (0,0,0)$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 4 & -3 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to 7R_3 - 6R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{7} R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_3 = t \\ c_2 = t \\ c_1 = -t \end{array}$$

If $t = 1$, then $(3,2,-3) = (1,3,1) - (-2,1,4)$

By the +/- theorem, if $B_W = \{(1,3,1),(-2,1,4)\}$, then $\text{span}(B_w) = \text{span}(B) = W$.
Since B_W is linearly independent (the two vectors are not multiples of each other), if forms a basis for *W*.
$\dim(W) = 2$

$$\vec{n} = (1,3,1)\times(-2,1,4) = \begin{vmatrix} i & j & k \\ 1 & 3 & 1 \\ -2 & 1 & 4 \end{vmatrix} = (11,-6,7)$$

Ergo, *W* is the plane $\pi: 11x - 6y + 7z = 0$.

c) $W = \{(2a-b+c, a+b+c, 3a+2c) : a,b,c \in \mathbb{R}\}$

Since $(2a-b+c, a+b+c, 3a+2c) = a(2,1,3) + b(-1,1,0) + c(1,1,2)$

then if $B = \{(2,1,3),(-1,1,0),(1,1,2)\}$ we have $W = \text{span}(B)$.

$c_1(2,1,3) + c_2(-1,1,0) + c_3(1,1,2) = (0,0,0)$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 0 \end{array}\right] \xrightarrow[R_3 \to 2R_3 - 3R_1]{R_2 \to 2R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{3}R_2]{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} c_3 = t \\ c_2 = \frac{-1}{3}t \\ c_1 = \frac{-2}{3}t \end{array}$$

If $t = 1$, then $(1,1,2) = \frac{2}{3}(2,1,3) + \frac{1}{3}(-1,1,0)$

By the +/- theorem, if $B_W = \{(2,1,3),(-1,1,0)\}$, then $\text{span}(B_w) = \text{span}(B) = W$.
Since B_W is linearly independent, if forms a basis for *W*.

23. Find a basis and the dimension of each of the following subspaces *W*,

a) $W = \left\{ \begin{bmatrix} a & a+b \\ a-b & b \end{bmatrix} : a,b \in \mathbb{R} \right\}$

Since $\begin{bmatrix} a & a+b \\ a-b & b \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ then

if $B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$ we have $W = \text{span}(B)$.

Since *B* is linearly independent (the two matrices are not multiples of each other) then *B* is a basis for *W* and $\dim(W) = 2$

b) $W=\left\{\begin{bmatrix} a+b+2c & 2a+3b+3c \\ 2a+3b+3c & -a+b-4c \end{bmatrix}: a,b,c\in\mathbb{R}\right\}$

Since $\begin{bmatrix} a+b+2c & 2a+3b+3c \\ 2a+3b+3c & -a+b-4c \end{bmatrix}=a\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}+b\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}+c\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}$ then

if $B=\left\{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix},\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix},\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}\right\}$ we have $W=\operatorname{span}(B)$.

$$c_1\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}+c_2\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}+c_3\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 3 & 3 & 0 \\ 2 & 3 & 3 & 0 \\ -1 & 1 & -4 & 0 \end{array}\right]\xrightarrow[R_4\to R_4+R_1]{\substack{R_2\to R_2-2R_1 \\ R_3\to R_3-2R_1}}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_4\to R_4-2R_2]{R_3\to R_3-R_2}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} c_3&=t \\ c_2&=t \\ c_1&=-3t \end{aligned}$$

If $t=1$, then $\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}=3\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}-\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, so if $B_W=\left\{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix},\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\right\}$, then by the +/- theorem $\operatorname{span}(B_W)=\operatorname{span}(B)=W$.

Since B is linearly independent (the two matrices are not multiples of each other) then B is a basis for W and $\dim(W)=2$

c) $W=\{ax^2+(b-a)x+b:\ a,b\in\mathbb{R}\}$

Since $ax^2+(b-a)x+b=a(x^2-x)+b(x+1)$ then if $B=\{x^2-1,x+1\}$ we have $W=\operatorname{span}(B)$. Since B is linearly independent (the two polynomials are not multiples of each other) then B is a basis for W and $\dim(W)=2$

24. Find all values of t for which S is linearly independent.

a) $S=\{(2,3,5),(-1,t,-1),(-1,-1,t)\}$

$$c_1(2,3,5)+c_2(-1,t,-1)+c_3(-1,-1,t)=(0,0,0)$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 3 & t & -1 & 0 \\ 5 & -1 & t & 0 \end{array}\right]\xrightarrow[R_3\to 2R_3-5R_1]{R_2\to 2R_2-3R_1}\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 2t+3 & 1 & 0 \\ 0 & 3 & 2t+5 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3 & 2t+5 & 0 \\ 0 & 2t+3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to 3R_3 - (2t+3)R_2} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3 & 2t+5 & 0 \\ 0 & 0 & -4t^2-16t-12 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 \to \frac{1}{3}R_1 \\ R_1 \to \frac{1}{-4t^2-16t-12}R_1}]{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{2t+5}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Illegal if $-4t^2 - 16t - 12 \neq 0$

$$t^2 + 4t + 3 = 0$$
$$(t+3)(t+1) = 0$$
$$t = -3, -1$$

Thus if $t \neq -3, -1$ then the solution is $c_3 = c_2 = c_1 = 0$ so S is linearly independent.

If $t = -3$, then $\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{3}R_2]{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad c_3 = t$, $c_2 = \frac{1}{3}t$, $c_1 = \frac{7}{6}t$

so S is linearly dependent

If $t = -1$, then $\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{3}R_2]{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\quad c_3 = t$, $c_2 = -t$, $c_1 = 0$

so S is linearly dependent

b) $S = \left\{3x^2 + x + 4, 2x^2 - x, x^2 + tx + 2t\right\}$

$c_1\left(3x^2 + x + 4\right) + c_2\left(2x^2 - x\right) + c_3\left(x^2 + tx + 2t\right) = 0$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 1 & -1 & t & 0 \\ 4 & 0 & 2t & 0 \end{array}\right] \xrightarrow[R_3 \to 3R_3 - 4R_1]{R_2 \to 3R_2 - R_1} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & -5 & 3t-1 & 0 \\ 0 & -8 & 6t-4 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to 5R_3 - 8R_2} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & -5 & 3t-1 & 0 \\ 0 & 0 & 6t-12 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to \frac{-1}{5}R_2 \\ R_3 \to \frac{1}{6t-12}R_3}]{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{3t-1}{-5} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Illegal if $6t-12=0$

$t=2$

Thus if $t\neq 2$, then the solution is $c_3=c_2=c_1=0$ so S is linearly independent.

If $t=2$ then $\left[\begin{array}{ccc|c}3&2&1&0\\0&-5&5&0\\0&0&0&0\end{array}\right]\xrightarrow[R_2\to\frac{-1}{5}R_2]{R_1\to\frac{1}{3}R_1}\left[\begin{array}{ccc|c}1&\frac{2}{3}&\frac{1}{3}&0\\0&1&-1&0\\0&0&0&0\end{array}\right]\quad \begin{array}{l}c_3=t\\c_2=t\\c_1=-t\end{array}$

thus S is linearly dependent.

25. Do the following sets S span V?

a) $S=\{(2,-4,1),(1,2,-3),(5,-14,6)\}\qquad V=\mathbb{R}^3$

Let $(a,b,c)\in\mathbb{R}^3$.

$$c_1(2,-4,1)+c_2(1,2,-3)+c_3(5,-14,6)=(a,b,c)$$

$$\left[\begin{array}{ccc|c}2&1&5&a\\-4&2&-14&b\\1&-3&6&c\end{array}\right]\xrightarrow[R_3\to 2R_3-R_1]{R_2\to R_2+2R_1}\left[\begin{array}{ccc|c}2&1&5&a\\0&4&-4&b+2a\\0&-7&7&2c-a\end{array}\right]$$

$$\xrightarrow{R_3\to 4R_3+7R_2}\left[\begin{array}{ccc|c}2&1&5&a\\0&4&-4&b+2a\\0&0&0&10a+7b+8c\end{array}\right]$$

There is no solution if $10a+7b+8c\neq 0$. Thus S does not span $\mathbb{R}^3$.

b) $S=\left\{\begin{bmatrix}2&-1\\-1&3\end{bmatrix},\begin{bmatrix}3&2\\2&1\end{bmatrix},\begin{bmatrix}4&1\\1&4\end{bmatrix}\right\}\qquad V=S_{2,2}$ (The set of symmetric 2×2 matrices)

Let $A=\begin{bmatrix}a&b\\b&c\end{bmatrix}\in S_{2,2}$

$$c_1\begin{bmatrix}2&-1\\-1&3\end{bmatrix}+c_2\begin{bmatrix}3&2\\2&1\end{bmatrix}+c_3\begin{bmatrix}4&1\\1&4\end{bmatrix}=\begin{bmatrix}a&b\\b&c\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}2&3&4&a\\-1&2&1&b\\3&1&4&c\end{array}\right]\xrightarrow[R_3\to 2R_3-3R_2]{R_2\to 2R_2+R_1}\left[\begin{array}{ccc|c}2&3&4&a\\0&7&6&a+2b\\0&-7&-4&-3a+2c\end{array}\right]$$

$$\xrightarrow{R_3\to R_3+R_2}\left[\begin{array}{ccc|c}2&3&4&a\\0&7&6&a+2b\\0&0&2&-2a+2b+2c\end{array}\right]\xrightarrow[R_3\to\frac{1}{2}R_3]{\substack{R_1\to\frac{1}{2}R_1\\R_2\to\frac{1}{7}R_2}}\left[\begin{array}{ccc|c}1&\frac{3}{2}&2&\frac{1}{2}a\\0&1&\frac{6}{7}&\frac{1}{7}a+\frac{2}{7}b\\0&0&1&-a+b+c\end{array}\right]$$

$$c_3=-a+b+c,\quad c_2=a-\tfrac{4}{7}b-\tfrac{6}{7}c,\quad c_1=a-\tfrac{8}{7}b-\tfrac{5}{7}c$$

Thus S spans $S_{2,2}$

c) $S=\{x^3-2x+1,x^3+x^2,x^3+x^2+x+1\}\qquad V=P_3$

Since $n(S) = 3 < \dim(P_3) = 4$, then S does not span P_3

26. Are the following sets S bases for the vector space V?

a) $S=\{(2,-1,3),(1,1,7),(-2,4,1)\}$, $V=\mathbb{R}^3$

$c_1(2,-1,3)+c_2(1,1,7)+c_3(-2,4,1)=(0,0,0)$

$$\left[\begin{array}{ccc|c}2&1&-2&0\\-1&1&4&0\\3&7&1&0\end{array}\right]\xrightarrow[R_3\to 2R_3-3R_1]{R_2\to 2R_2+R_1}\left[\begin{array}{ccc|c}2&1&-2&0\\0&3&6&0\\0&11&8&0\end{array}\right]$$

$$\xrightarrow{R_3\to 3R_3-11R_2}\left[\begin{array}{ccc|c}2&1&-2&0\\0&3&6&0\\0&0&-42&0\end{array}\right]\xrightarrow[R_3\to \frac{-1}{42}R_3]{R_1\to\frac{1}{2}R_1,\ R_2\to\frac{1}{3}R_2}\left[\begin{array}{ccc|c}1&\frac{1}{2}&-1&0\\0&1&2&0\\0&0&1&0\end{array}\right]$$

$c_1=c_2=c_3=0$

Since S is linearly independent and $n(S)=3=\dim(\mathbb{R}^3)$, then S is a basis for $\mathbb{R}^3$

b) $S=\{(-3,5,1),(2,-7,12)\}$, $V=\mathbb{R}^3$

No since $n(S)=2\neq\dim(\mathbb{R}^3)=3$.

c) $S=\{x^2+1,x^2-1,x^2+x+1,x^2-x-1\}$, $V=P_2$?

No since $n(S)=4\neq\dim(P_2)=3$

d) $S=\left\{\begin{bmatrix}2&-1\\2&3\end{bmatrix},\begin{bmatrix}4&-1\\4&3\end{bmatrix},\begin{bmatrix}3&3\\-1&1\end{bmatrix},\begin{bmatrix}5&5\\1&-2\end{bmatrix}\right\}$, $V=M_{2,2}$?

$$c_1\begin{bmatrix}2&-1\\2&3\end{bmatrix}+c_2\begin{bmatrix}4&-1\\4&3\end{bmatrix}+c_3\begin{bmatrix}3&3\\-1&1\end{bmatrix}+c_4\begin{bmatrix}5&5\\1&-2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}2&4&3&5&0\\-1&-1&3&5&0\\2&4&-1&1&0\\3&3&1&-2&0\end{array}\right]\xrightarrow[R_4\to 2R_4-3R_1]{R_2\to 2R_2+R_1,\ R_3\to R_3-R_1}\left[\begin{array}{cccc|c}2&4&3&5&0\\0&2&9&15&0\\0&0&-4&-4&0\\0&-6&-7&-19&0\end{array}\right]$$

$$\xrightarrow{R_4\to R_4+3R_2}\left[\begin{array}{cccc|c}2&4&3&5&0\\0&2&9&15&0\\0&0&-4&-4&0\\0&0&20&26&0\end{array}\right]\xrightarrow{R_4\to R_4+5R_3}\left[\begin{array}{cccc|c}2&4&3&5&0\\0&2&9&15&0\\0&0&-4&-4&0\\0&0&0&6&0\end{array}\right]$$

$$\xrightarrow[R_4\to\frac{1}{6}R_4]{R_1\to\frac{1}{2}R_1,\ R_2\to\frac{1}{2}R_2,\ R_3\to\frac{-1}{4}R_3}\left[\begin{array}{cccc|c}1&2&\frac{3}{2}&\frac{5}{2}&0\\0&1&\frac{9}{2}&\frac{15}{2}&0\\0&0&1&1&0\\0&0&0&1&0\end{array}\right]\qquad c_4=c_3=c_2=c_1=0$$

Since S is linearly independent and $n(S)=4=\dim(M_{2,2})$, then S is a basis for $M_{2,2}$

27. Add or subtract vectors to the set S to so that it forms a basis for $\mathbb{R}^3$

a) $S=\{(1,3,5),(2,-1,3)\}$

If $\vec{v}=(1,0,0)$, then

$$c_1(1,3,5)+c_2(2,-1,3)=(1,0,0)$$

$$\left[\begin{array}{cc|c}1&2&1\\3&-1&0\\5&3&0\end{array}\right]\xrightarrow[R_3\to R_3-5R_1]{R_2\to R_2-3R_1}\left[\begin{array}{cc|c}1&2&1\\0&-7&-3\\0&-7&-5\end{array}\right]\xrightarrow{R_3\to R_3-R_2}\left[\begin{array}{cc|c}1&2&1\\0&-7&-3\\0&0&-2\end{array}\right]$$

No solution, so $(1,0,0)\notin\operatorname{span}(S)$

Since S is linearly independent (the two vectors are not multiples of each other), then by the +/- theorem $B=\{(1,3,5),(2,-1,3),(1,0,0)\}$ is linearly independent.

Since $n(B)=3=\dim(\mathbb{R}^3)$, then B is a basis for $\mathbb{R}^3$

b) $S=\{(1,2,4),(1,3,9),(1,0,-6)\}$

Let us verify independence.

$$c_1(1,2,4)+c_2(1,3,9)+c_3(1,0,-6)=(0,0,0)$$

$$\left[\begin{array}{ccc|c}1&1&1&0\\2&3&0&0\\4&9&-6&0\end{array}\right]\xrightarrow[R_3\to R_3-4R_1]{R_2\to R_2-2R_1}\left[\begin{array}{ccc|c}1&1&1&0\\0&1&-2&0\\0&5&-10&0\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-5R_1}\left[\begin{array}{ccc|c}1&1&1&0\\0&1&-2&0\\0&0&0&0\end{array}\right]\qquad \begin{aligned}c_3&=t\\c_2&=2t\\c_1&=-3t\end{aligned}$$

If $t=1$, then $(1,0,-6)=3(1,2,4)-2(1,3,9)$

By the +/- theorem, if $B=\{(1,2,4),(1,3,9)\}$, then $\operatorname{span}(S)=\operatorname{span}(B)$.

If $\vec{v}=(1,0,0)$, then

$$c_1(1,2,4)+c_2(1,3,9)=(1,0,0)$$

$$\left[\begin{array}{cc|c}1&1&1\\2&3&0\\4&9&0\end{array}\right]\xrightarrow[R_3\to R_3-4R_1]{R_2\to R_2-2R_1}\left[\begin{array}{cc|c}1&2&1\\0&1&-2\\0&5&-4\end{array}\right]\xrightarrow{R_3\to R_3-5R_2}\left[\begin{array}{cc|c}1&2&1\\0&1&-2\\0&0&6\end{array}\right]$$

No solution, so $(1,0,0)\notin\operatorname{span}(B)$

Since B is linearly independent (the two vectors are not multiples of each other), then by the +/- theorem $B_S=\{(1,2,4),(1,3,9),(1,0,0)\}$ is linearly independent.

Since $n(B_S)=3=\dim(\mathbb{R}^3)$, then B_S is a basis for $\mathbb{R}^3$.

28. Find a basis for $\text{span}(S)$ if

a) $S = \{(-1,1,-1),(2,1,1),(1,5,1)\}$

$$c_1(-1,1,-1)+c_2(2,1,1)+c_3(1,5,1)=(0,0,0)$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 1 & 1 & 5 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to 3R_3 + R_2} \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{6}R_3]{R_1 \to -R_1,\ R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{array}$$

Since S is linearly independent, then it is a basis for $\text{span}(S)$. Since $\dim(\text{span}(S)) = 3 = \dim(\mathbb{R}^3)$, then $\text{span}(S) = \mathbb{R}^3$

b) $S = \{(1,3,-2),(2,6,-4),(-3,-9,6)\}$

$$c_1(1,3,-2)+c_2(2,6,-4)+c_3(-3,-9,6)=(0,0,0)$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 3 & 6 & -9 & 0 \\ -2 & -4 & 6 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} c_3 = t \\ c_2 = s \\ c_2 = -2s + 3t \end{array}$$

If $t = 1$ and $s = 0$ then $(-3,-9,6) = -3(1,3,-2)$

If $t = 0$ and $s = 1$ then $(2,6,-4) = 2(1,3,-2)$

By the +/- theorem, if $B = \{(1,3,-2)\}$, then $\text{span}(B) = \text{span}(S)$ and since B is linearly independent (only a single vector) then B is linearly independent, thus B is a basis for $\text{span}(S)$.

Geometrically, $\text{span}(S)$ is the line $x = \frac{y}{3} = \frac{z}{-2}$.

c) $S = \{(1,2,-1),(2,3,1),(4,7,-1),(1,1,2)\}$

$$c_1(1,2,-1)+c_2(2,3,1)+c_3(4,7,-1)+c_4(1,1,2)=(0,0,0)$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 0 \\ 2 & 3 & 7 & 1 & 0 \\ -1 & 1 & -1 & 2 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 3 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} c_4 = t \\ c_3 = s \\ c_2 = -t - s \\ c_1 = t - 2s \end{array}$$

If $t = 1$ and $s = 0$ then $(1,1,2) = -(1,2,-1) + (2,3,1)$

If $t = 0$ and $s = 1$ then $(4,7,-1) = 2(1,2,-1) + (2,3,1)$

By the +/- theorem, if $B = \{(1,2,-1),(2,3,1)\}$, then $\text{span}(B) = \text{span}(S)$ and since B is linearly independent (the two vectors are not multiples of each other), thus B is a basis for $\text{span}(S)$.

$$\vec{n} = (1,2,-1) \times (2,3,1) = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix} = (5,-3,-1)$$

Geometrically, $\text{span}(S)$ is the line $5x - 3y - z = 0$.

d) $S = \{(1,2,3,4),(4,3,2,1),(1,1,1,1)\}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 5R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{-1}{5} R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for $span(S)$ is $B_S = \{(1,2,3,4),(0,1,2,3)\}$

Note: If you used the +/- theorem, then you obtain $B_S = \{(1,2,3,4),(4,3,2,1)\}$

29. Find the coordinate vector for $\vec{w}$ in the vector space V relative to the basis S.

a) $\vec{w} = (-5,-6,24)$, $V = \mathbb{R}^3$ and $S = \{(2,-1,4),(1,2,4),(-3,-3,4)\}$.

$c_1(2,-1,4) + c_2(1,2,4) + c_3(-3,-3,4) = (-5,-6,24)$

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & -5 \\ -1 & 2 & -3 & -6 \\ 4 & 4 & 4 & 24 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to 2R_2 + R_1} \left[\begin{array}{ccc|c} 2 & 1 & -3 & -5 \\ 0 & 5 & -9 & -17 \\ 0 & 2 & 10 & 34 \end{array}\right]$$

$$\xrightarrow{R_3 \to 5R_3 - 2R_2} \left[\begin{array}{ccc|c} 2 & 1 & -3 & -5 \\ 0 & 5 & -9 & -17 \\ 0 & 0 & 68 & 204 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{68} R_3]{R_1 \to \frac{1}{2} R_1,\ R_2 \to \frac{1}{5} R_2} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{-3}{2} & \frac{-5}{2} \\ 0 & 1 & \frac{-9}{5} & \frac{-17}{5} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$c_3 = 3,\ c_2 = 2,\ c_1 = 1$

$\vec{w}_S = (1,2,3)$

b) $\vec{w}=(12,-7,10)$, $V=\mathbb{R}^3$ and $S=\{(3,-1,5),(1,2,3),(2,-1,-1)\}$.

$c_1(3,-1,5)+c_2(1,2,3)+c_3(2,-1,-1)=(12,-7,10)$

$$\left[\begin{array}{ccc|c}3&1&2&12\\-1&2&-1&-7\\5&3&-1&10\end{array}\right]\xrightarrow[R_3\to 3R_3-5R_1]{R_2\to 3R_2+R_1}\left[\begin{array}{ccc|c}3&1&2&12\\0&7&-1&-9\\0&4&-13&-30\end{array}\right]$$

$$\xrightarrow{R_3\to 7R_3-4R_1}\left[\begin{array}{ccc|c}3&1&2&12\\0&7&-1&-9\\0&0&-87&-174\end{array}\right]\xrightarrow[R_3\to\frac{-1}{87}R_3]{R_1\to\frac{1}{3}R_1,\ R_2\to\frac{1}{7}R_2}\left[\begin{array}{ccc|c}1&\frac{1}{3}&\frac{2}{3}&4\\0&1&\frac{-1}{7}&\frac{-9}{7}\\0&0&1&2\end{array}\right]$$

$c_3=2,\ c_2=-1,\ c_1=3$

$\vec{w}_S=(3,-1,2)$

c) $\vec{w}=(4,-3,5,3)$, $V=\text{Span}(\{(3,-1,4,0),(0,1,-2,4),(2,2,1,1)\})$ and $S=\{(3,-1,4,0),(0,1,-2,4),(2,2,1,1)\}$.

$c_1(3,-1,4,0)+c_2(0,1,-2,4)+c_3(2,2,1,1)=(4,-3,5,3)$

$$\left[\begin{array}{ccc|c}3&0&2&4\\-1&1&2&-3\\4&-2&1&5\\0&4&1&3\end{array}\right]\xrightarrow[R_3\to 3R_3-4R_1]{R_2\to 3R_2+R_1}\left[\begin{array}{ccc|c}3&0&2&4\\0&3&8&-5\\0&-6&-5&-1\\0&4&1&3\end{array}\right]$$

$$\xrightarrow[R_4\to 3R_4-4R_2]{R_3\to R_3+2R_2}\left[\begin{array}{ccc|c}3&0&2&4\\0&3&8&-5\\0&0&11&-11\\0&0&-29&29\end{array}\right]\xrightarrow{R_4\to 11R_4+29R_3}\left[\begin{array}{ccc|c}3&0&2&4\\0&3&8&-5\\0&0&11&-11\\0&0&0&0\end{array}\right]$$

$$\xrightarrow[R_3\to\frac{1}{11}R_3]{R_1\to\frac{1}{3}R_1,\ R_2\to\frac{1}{3}R_2}\left[\begin{array}{ccc|c}1&0&\frac{2}{3}&\frac{4}{3}\\0&1&\frac{8}{3}&\frac{-5}{3}\\0&0&1&-1\\0&0&0&0\end{array}\right]\qquad \begin{array}{l}c_3=-1\\c_2=1\\c_1=2\end{array}\qquad \vec{w}_S=(2,1,-1)$$

d) $\vec{w}=9x^2-x+11$, $V=P_2$ and $S=\{x^2+x,x^2-1,2x^2-3x+4\}$.

$c_1(x^2+x)+c_2(x^2-1)+c_3(2x^2-3x+4)=9x^2-x+11$

$$\left[\begin{array}{ccc|c}1&1&2&9\\1&0&-3&-1\\0&-1&4&11\end{array}\right]\xrightarrow{R_2\to R_2-R_1}\left[\begin{array}{ccc|c}1&1&1&9\\0&-1&-5&-10\\0&-1&4&11\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-R_2}\left[\begin{array}{ccc|c}1&1&2&9\\0&-1&-5&-10\\0&0&9&21\end{array}\right]\xrightarrow[R_3\to\frac{1}{9}R_3]{R_2\to -R_2}\left[\begin{array}{ccc|c}1&1&2&9\\0&1&5&10\\0&0&1&\frac{7}{3}\end{array}\right]\qquad \begin{array}{l}c_3=\frac{7}{3}\\c_2=\frac{-5}{3}\\c_1=6\end{array}$$

$\vec{w}_S=\left(6,\frac{-5}{3},\frac{7}{3}\right)$

30. Find a basis, and the dimension, for the solution space of $AX = 0$.

a)
$$\begin{aligned} x - 2y + z - w &= 0 \\ 3x + y + w &= 0 \\ 4x + 6y - 2z + 4w &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 6 & -2 & 4 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 7 & -3 & 4 & 0 \\ 0 & 14 & -6 & 8 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 7 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{7}R_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & \frac{-3}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$w = t, \quad z = s, \quad y = \tfrac{3}{7}s - \tfrac{4}{7}t, \quad x = \tfrac{-1}{7}s - \tfrac{1}{7}t$$

$$(x, y, z, t) = \left(\tfrac{-1}{7}s - \tfrac{1}{7}t, \tfrac{3}{7}s - \tfrac{4}{7}t, s, t\right) = s\left(\tfrac{-1}{7}, \tfrac{2}{7}, 1, 0\right) + t\left(\tfrac{-1}{7}, \tfrac{-4}{7}, 0, 1\right)$$

$$B_{SS} = \left\{\left(\tfrac{-1}{7}, \tfrac{3}{7}, 1, 0\right), \left(\tfrac{-1}{7}, \tfrac{-4}{7}, 0, 1\right)\right\} \qquad \dim(SS) = 2$$

b)
$$\begin{aligned} x - y + 3z &= 0 \\ 2x + 5y + 6z &= 0 \\ x - 8y + 3z &= 0 \\ 2x + y + 6z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & -8 & 3 & 0 \\ 2 & 1 & 6 & 0 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_4}]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \to 4R_4 - R_2]{R_3 \to 8R_3 + 7R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{8}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$z = t, \quad y = 0, \quad x = -3t$$

$$(x, y, z) = t(-3, 0, 1)$$

$$B_{SS} = \{(-3, 0, 1)\} \quad \dim(SS) = 1$$

31. Find the minimum or maximum values of the given objective function, subject to the indicated constraints.

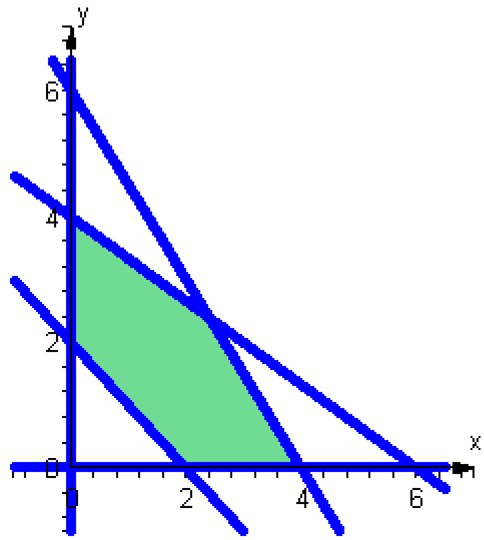
a) Objective function: $f = 3x + 5y$

Constraints: $x + y \geq 2$

$2x + 3y \leq 12$

$3x + 2y \leq 12$

$x \geq 0,\ y \geq 0$



At $A(0,2)$ $f = 10$

$B(0,4)$ $f = 20$

$C\left(\frac{12}{5},\frac{12}{5}\right)$ $f = \frac{96}{5}$

$D(4,0)$ $f = 12$

$E(2,0)$ $f = 6$

Thus the minimum value of f is 6 when $x = 2$ and $y = 0$, and the maximum value is 20 when $x = 0$ and $y = 4$.

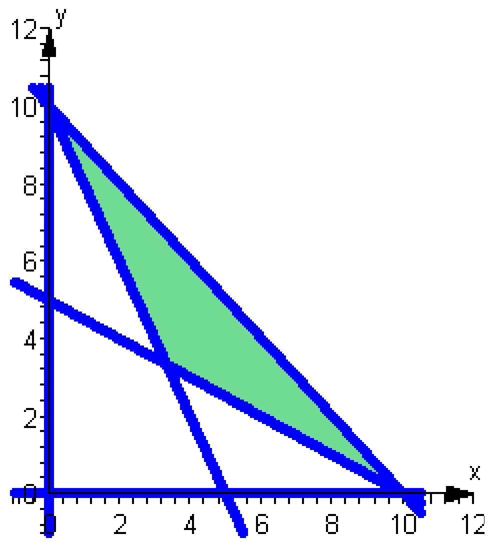
b) Objective function: $f = 5x + 2y$

Constraints: $x + y \leq 10$

$2x + y \geq 10$

$x + 2y \geq 10$

$x \geq 0,\ y \geq 0$



At $A(0,10)$ $f = 20$

$B(10,0)$ $f = 50$

$C\left(\frac{10}{3},\frac{10}{3}\right)$ $f = \frac{70}{3}$

Thus the minimum value of f is 20 when $x = 0$ and $y = 10$, and the maximum value is 50 when $x = 10$ and $y = 0$.

c) Objective function: $z = 2x + 5y$

Constraints: $2x + y \ge 8$

$-4x + y \le 2$

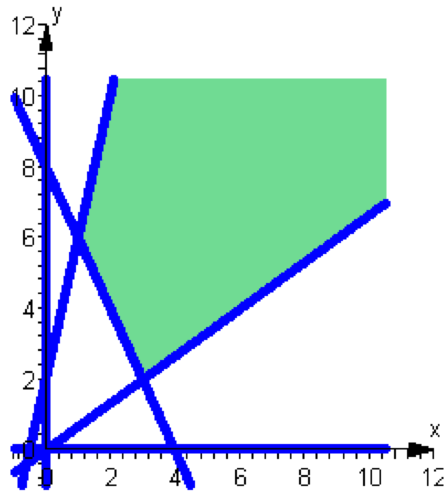
$2x - 3y \le 0$

$x \ge 0,\ y \ge 0$

At $A(1,6)$ $f = 32$

$B(3,2)$ $f = 16$

Thus there is no maximum, and the minimum value is 16 when $x = 3$ and $y = 2$



32. An entrepreneur is having a design group produce at least six samples of a new kind of fastener that he wants to market. It cost \$9.00 to produce each metal fastener and \$4.00 to produce each plastic fastener. He wants to have at least two of each version of the fastener and needs to have all the samples 24 hours from now. It takes 4 hours to produce each metal sample and 2 hours to produce each plastic sample. To minimize the cost of the samples, how many of each kind should the entrepreneur order? What will be the cost of the samples?

Minimize $C = 9x + 4y$

Constraints: $x + y \ge 6$

$x \ge 2$

$y \ge 2$

$4x + 2y \le 24$

At $A(2,4)$ $C = 34$

$B(2,8)$ $C = 50$

$C(5,2)$ $C = 53$

$D(4,2)$ $D = 44$

Thus minimal cost of the samples is \$34 when 2 metal and 4 plastic samples are ordered.