

MATHEMATICS 201-105-RE

Linear Algebra

Martin Huard

Winter 2006

Properties of Vectors

Properties of Vector Sum

Let V be the set of all algebraic vectors having the same dimension and let $\vec{u}, \vec{v}, \vec{w} \in V$.

1. $\vec{u} + \vec{v} \in V$ closure under addition
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ commutativity
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ associativity
4. $\vec{0} + \vec{u} = \vec{u}$
5. $\vec{u} + (-\vec{u}) = \vec{0}$

Properties of scalar multiplication

Let V be the set of all algebraic vectors having the same dimension with $\vec{u}, \vec{v} \in V$ and k, l scalars.

6. $k\vec{u} \in V$ closure under scalar multiplication
7. $(k+l)\vec{u} = k\vec{u} + l\vec{u}$ distributivity
8. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ distributivity
9. $k(l\vec{u}) = (kl)\vec{u}$ associativity
10. $1\vec{u} = \vec{u}$

Definition of the dot product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Properties of the dot product

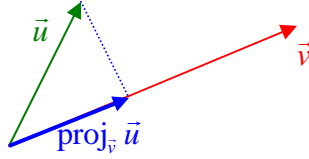
If \vec{u}, \vec{v} and \vec{w} are vectors in \mathbb{R}^n and c is a scalar, then we have the following properties.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$
5. $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$
6. $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$ (Cauchy Scharwz Inequality)
7. $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$ (Defintion of the angle between two vectors)

Orthogonal Projections

If \vec{u} and \vec{v} are vectors in \mathbb{R}^n such that $\vec{v} \neq 0$, then the orthogonal projection of \vec{u} onto \vec{v} is given by

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$



Definition of the cross product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2) \hat{\mathbf{i}} + (u_3 v_1 - u_1 v_3) \hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1) \hat{\mathbf{k}}$$

Properties of the Cross Product

Let \vec{u} , \vec{v} and \vec{w} be vectors in \mathbb{R}^3 and k a scalar.

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
5. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{u} = \vec{0}$
7. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$

Relationships involving Cross and Dot Products

Let \vec{u} , \vec{v} and \vec{w} be vectors in \mathbb{R}^3 .

1. $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ $\vec{u} \perp \vec{u} \times \vec{v}$
2. $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$ $\vec{v} \perp \vec{u} \times \vec{v}$
3. $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$
4. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{u} \cdot \vec{v}) \vec{w}$
5. $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{v}) \vec{w} - (\vec{v} \cdot \vec{w}) \vec{u}$
6. $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{w} \times \vec{u})$