MATHEMATICS 201-105-RE
Linear Algebra
Martin Huard
Winter 2006

IX - Dot Product

1. Find (i) $\vec{u} \bullet \vec{v}$ (ii) $\vec{u} \bullet \vec{u}$ (iii) $\|\vec{u}\|$ and (iv) $(\vec{u} \bullet \vec{v})\vec{v}$
    a) $\vec{u} = (3,4)$, $\vec{v} = (2,-3)$
    b) $\vec{u} = (1,-2,3)$, $\vec{v} = (4,5-1)$
    c) $\vec{u} = (4,0,-3,5)$, $\vec{v} = (0,2,5,4)$
    d) $\vec{u} = (0,1,-3,4,1)$, $\vec{v} = (-4,5,1,2,2)$

2. Find the angle θ between the given vectors.
    a) $\vec{u} = (3,4)$, $\vec{v} = (-4,3)$
    b) $\vec{u} = (1,-2,3)$, $\vec{v} = (4,5-1)$
    c) $\vec{u} = (4,0,-3,5)$, $\vec{v} = (-8,0,6,-10)$
    d) $\vec{u} = (0,1,-3,4,1)$, $\vec{v} = (-4,5,1,2,2)$

3. Find all vectors $\vec{w}$ that are perpendicular to $\vec{u}$.
    a) $\vec{u} = (-1,5)$
    b) $\vec{u} = (3,0)$
    c) $\vec{u} = (-1,1,2)$
    d) $\vec{u} = (0,1,0,0)$

4. Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal, parallel or neither.
    a) $\vec{u} = (3,0)$, $\vec{v} = (1,1)$
    b) $\vec{u} = (\frac{1}{2},-\frac{2}{3})$, $\vec{v} = (4,3)$
    c) $\vec{u} = (4,0,-3)$, $\vec{v} = (0,-2,5)$
    d) $\vec{u} = (1,-3,2)$, $\vec{v} = (-1,-1,-1)$
    e) $\vec{u} = (4,-5,2,6)$, $\vec{v} = (-2,\frac{5}{2},-1,-3)$

5. Verify the Cauchy-Schwarz Inequality and the Triangular Inequality for the given vectors.
    a) $\vec{u} = (3,0)$, $\vec{v} = (1,1)$
    b) $\vec{u} = (-3,2)$, $\vec{v} = (4,3)$
    c) $\vec{u} = (2,2,2)$, $\vec{v} = (0,-2,1)$

6. Determine if the triangle ABC is a right triangle, and if so, find at which point the right angle is.
   a) $A(1,0)$, $B(2,3)$ and $C(6,0)$
   b) $A(6,5)$, $B(1,3)$ and $C(3,-2)$
   c) $A(1,2,3)$, $B(-1,-3,2)$ and $C(5,0,5)$

7. Prove that if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^n$, then
$$\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$$

8. Prove that if $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then
$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

9. Prove that if $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then
$$\vec{u} \bullet \vec{v} = \tfrac{1}{4}\|\vec{u} + \vec{v}\|^2 - \tfrac{1}{4}\|\vec{u} - \vec{v}\|^2$$

10. Prove that the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.

11. Let ABCD be a parallelogram. Prove that
$$\left\|\overrightarrow{AC}\right\|^2 + \left\|\overrightarrow{BD}\right\|^2 = \left\|\overrightarrow{AB}\right\|^2 + \left\|\overrightarrow{BC}\right\|^2 + \left\|\overrightarrow{CD}\right\|^2 + \left\|\overrightarrow{DA}\right\|^2$$

12. Find the projection of (i) $\vec{u}$ onto $\vec{v}$ (ii) $\vec{v}$ onto $\vec{u}$
   a) $\vec{u} = (3,4)$, $\vec{v} = (2,-3)$
   b) $\vec{u} = (1,-2,3)$, $\vec{v} = (4,5-1)$
   c) $\vec{u} = (4,0,-3,5)$, $\vec{v} = (0,2,5,4)$
   d) $\vec{u} = (0,1,-3,4,1)$, $\vec{v} = (-4,5,1,2,2)$

Answers

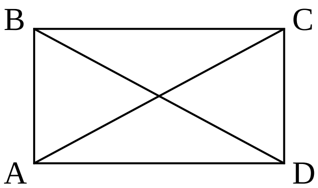
1. a) i. –6 ii. 25 iii. 5 iv. $(-12,18)$
   b) i. –9 ii. 14 iii. $\sqrt{14}$ iv. $(-36,-45,9)$
   c) i. 5 ii. 50 iii. $5\sqrt{2}$ iv. $(0,10,25,20)$
   d) i. 12 ii. 27 iii. $3\sqrt{3}$ iv. $(-48,60,12,24,24)$
2. a) 90° b) 111.8° c) 180° d) 70.9°
3. a) $t(5,1)$ b) $t(0,1)$ c) $s(1,1,0)+t(2,0,1)$ d) $r(1,0,0,0)+s(0,0,1,0)+t(0,0,0,1)$
4. a) neither b) orthogonal c) neither d) orthogonal e) parallel
6. a) No b) Yes in B c) Yes in A
7. Let $\vec{u}=(u_1,u_2,\ldots,u_n)$, $\vec{v}=(v_1,v_2,\ldots,v_n)$ and $\vec{w}=(w_1,w_2,\ldots,w_n)$.

$$\begin{aligned}\vec{u}\bullet(\vec{v}+\vec{w}) &= (u_1,u_2,\ldots,u_n)\bullet(v_1+w_1,v_2+w_2,\ldots,v_n+w_n)\\ &= u_1(v_1+w_1)+u_2(v_2+w_2)+\ldots+u_n(v_n+w_n)\\ &= u_1v_1+u_1w_1+u_2v_2+u_2w_2+\ldots+u_nv_n+u_nw_n\\ &= (u_1v_1+u_2v_2+\ldots+u_nv_n)+(u_1w_1+u_2w_2+\ldots+u_nw_n)\\ &= \vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}\end{aligned}$$

8.
$$\begin{aligned}\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v})+(\vec{u}-\vec{v})\bullet(\vec{u}-\vec{v})\\ &= \vec{u}\bullet\vec{u}+2\vec{u}\bullet\vec{v}+\vec{v}\bullet\vec{v}+\vec{u}\bullet\vec{u}-2\vec{u}\bullet\vec{v}+\vec{v}\bullet\vec{v}\\ &= 2\|\vec{u}\|^2+2\|\vec{v}\|^2\end{aligned}$$

9.
$$\begin{aligned}\tfrac{1}{4}\|\vec{u}+\vec{v}\|^2-\tfrac{1}{4}\|\vec{u}-\vec{v}\|^2 &= \tfrac{1}{4}(\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v})-\tfrac{1}{4}(\vec{u}-\vec{v})\bullet(\vec{u}-\vec{v})\\ &= \tfrac{1}{4}\vec{u}\bullet\vec{u}+\tfrac{1}{2}\vec{u}\bullet\vec{v}+\tfrac{1}{4}\vec{v}\bullet\vec{v}-\tfrac{1}{4}\vec{u}\bullet\vec{u}+\tfrac{1}{2}\vec{u}\bullet\vec{v}-\tfrac{1}{4}\vec{v}\bullet\vec{v}\\ &= \vec{u}\bullet\vec{v}\end{aligned}$$

10.

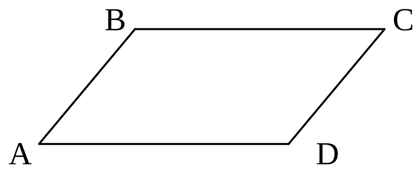


The diagonals are perpendicular if and only if $\overrightarrow{AC}\bullet\overrightarrow{BD}=0$

$$\begin{aligned}\overrightarrow{AC}\bullet\overrightarrow{BD} &= (\overrightarrow{AB}+\overrightarrow{BC})\bullet(\overrightarrow{BC}+\overrightarrow{CD})\\ &= \overrightarrow{AB}\bullet\overrightarrow{BC}+\overrightarrow{AB}\bullet\overrightarrow{CD}+\overrightarrow{BC}\bullet\overrightarrow{BC}+\overrightarrow{BC}\bullet\overrightarrow{CD}\\ &= 0+\overrightarrow{AB}\bullet\overrightarrow{CD}+\overrightarrow{BC}\bullet\overrightarrow{BC}+0 \quad \text{Since ABCD is a rectangle, } \overrightarrow{AB}\perp\overrightarrow{BC} \text{ and } \overrightarrow{BC}\perp\overrightarrow{CD}\\ &= -\overrightarrow{AB}\bullet\overrightarrow{AB}+\overrightarrow{BC}\bullet\overrightarrow{BC} \quad \text{Since } \overrightarrow{CD}=-\overrightarrow{AB}\\ &= -\|\overrightarrow{AB}\|^2+\|\overrightarrow{BC}\|^2\end{aligned}$$

Thus $\overrightarrow{AC}\bullet\overrightarrow{BD}=-\|\overrightarrow{AB}\|^2+\|\overrightarrow{BC}\|^2=0$ if and only if $\|\overrightarrow{AB}\|=\|\overrightarrow{BC}\|$, hence if and only if ABCD is a square.

11.



$$\left\|\overrightarrow{AC}\right\|^2 + \left\|\overrightarrow{BD}\right\|^2 = \left\|\overrightarrow{AB} + \overrightarrow{BC}\right\|^2 + \left\|\overrightarrow{BC} + \overrightarrow{CD}\right\|^2$$
$$= \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \bullet \left(\overrightarrow{AB} + \overrightarrow{BC}\right) + \left(\overrightarrow{BC} + \overrightarrow{CD}\right) \bullet \left(\overrightarrow{BC} \bullet \overrightarrow{CD}\right)$$
$$= \overrightarrow{AB} \bullet \overrightarrow{AB} + 2\overrightarrow{AB} \bullet \overrightarrow{BC} + \overrightarrow{BC} \bullet \overrightarrow{BC} + \overrightarrow{BC} \bullet \overrightarrow{BC} + 2\overrightarrow{BC} \bullet \overrightarrow{CD} + \overrightarrow{CD} \bullet \overrightarrow{CD}$$
$$= \left\|\overrightarrow{AB}\right\|^2 + 2\left\|\overrightarrow{BC}\right\|^2 + \left\|\overrightarrow{CD}\right\|^2 + 2\overrightarrow{AB} \bullet \overrightarrow{BC} + 2\overrightarrow{BC} \bullet \overrightarrow{CD}$$
$$= \left\|\overrightarrow{AB}\right\|^2 + \left\|\overrightarrow{BC}\right\|^2 + \left\|\overrightarrow{BC}\right\|^2 + \left\|\overrightarrow{CD}\right\|^2 + 2\overrightarrow{BC} \bullet \left(\overrightarrow{AB} + \overrightarrow{CD}\right)$$
$$= \left\|\overrightarrow{AB}\right\|^2 + \left\|\overrightarrow{BC}\right\|^2 + \left\|\overrightarrow{AD}\right\|^2 + \left\|\overrightarrow{CD}\right\|^2 + 2\overrightarrow{BC} \bullet \left(\overrightarrow{AB} + \overrightarrow{BA}\right) \text{ since } \overrightarrow{BC} = \overrightarrow{AD} \text{ and } \overrightarrow{CD} = \overrightarrow{BA}$$
$$= \left\|\overrightarrow{AB}\right\|^2 + \left\|\overrightarrow{BC}\right\|^2 + \left\|\overrightarrow{AD}\right\|^2 + \left\|\overrightarrow{CD}\right\|^2$$

12. a) $\text{proj}_{\vec{v}}\vec{u} = \left(\frac{-12}{13}, \frac{18}{13}\right)$ $\text{proj}_{\vec{u}}\vec{v} = \left(\frac{-18}{25}, \frac{-24}{25}\right)$

b) $\text{proj}_{\vec{v}}\vec{u} = \left(\frac{-6}{7}, \frac{-15}{14}, \frac{3}{14}\right)$ $\text{proj}_{\vec{u}}\vec{v} = \left(\frac{-9}{14}, \frac{9}{7}, \frac{-27}{14}\right)$

c) $\text{proj}_{\vec{v}}\vec{u} = \left(0, \frac{2}{9}, \frac{5}{9}, \frac{4}{9}\right)$ $\text{proj}_{\vec{u}}\vec{v} = \left(\frac{2}{5}, 0, \frac{-3}{10}, \frac{1}{2}\right)$

d) $\text{proj}_{\vec{v}}\vec{u} = \left(\frac{-24}{25}, \frac{6}{5}, \frac{12}{25}, \frac{12}{25}\right)$ $\text{proj}_{\vec{u}}\vec{v} = \left(0, \frac{4}{9}, \frac{-4}{3}, \frac{16}{9}, \frac{4}{9}\right)$