MATHEMATICS 201-105-RE
Linear Algebra
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Winter 2006

IV - Determinants

1. Let $A=\begin{bmatrix}3&-1&2\\5&7&-2\\1&4&6\end{bmatrix}$. Find

   a) M_{21} b) m_{21} c) c_{21} d) M_{33} e) m_{33} f) c_{33}

2. Find the determinant of the following matrices. Make sure you try both methods, cofactor expansion and elementary row operations.

   a) $\begin{bmatrix}5&9\\-1&3\end{bmatrix}$ b) $\begin{bmatrix}5&2\\3&-7\end{bmatrix}$ c) $\begin{bmatrix}1&-5&2\\6&-3&1\\4&6&3\end{bmatrix}$

   d) $\begin{bmatrix}-2&0&3\\1&10&-3\\1&4&5\end{bmatrix}$ e) $\begin{bmatrix}1&-4&0\\3&5&2\\0&3&9\end{bmatrix}$ f) $\begin{bmatrix}2&-1&3\\0&0&5\\2&-2&4\end{bmatrix}$

   g) $\begin{bmatrix}3&6&-5&4\\-2&0&6&0\\1&1&2&2\\0&3&-1&-1\end{bmatrix}$ h) $\begin{bmatrix}3&9&-2&1\\4&5&-8&3\\0&0&0&0\\8&-6&1&3\end{bmatrix}$ i) $\begin{bmatrix}5&2&0&0&-2\\0&1&4&3&2\\0&0&2&6&3\\0&0&3&4&1\\0&0&0&0&2\end{bmatrix}$

3. Consider the matrices $A=\begin{bmatrix}1&-7&5\\3&-2&2\\4&0&1\end{bmatrix}$ and $B=\begin{bmatrix}1&3&-2\\4&0&-3\\-7&5&4\end{bmatrix}$.

   Evaluate, using the properties of the determinant if possible.

   a) $\det(A)$ b) $\det(B)$ c) $\det(AB)$

   d) $\det(A^{-1})$ e) $\det(A^T)$ f) $\det(A+B)$.

4. Let A and B be 3×3 matrices such that $\det(A)=4$ and $\det(B)=-3$. Evaluate the following.

   a) $\det(AB)$ b) $\det\left(A^{-1}\right)$ c) $\det(A^T)$

   d) $\det(4B)$ e) $\det\left((AB)^T\right)$ f) $\det\left((AB)^{-1}\right)$

   g) $\det\left(B^6\right)$ h) $\det\left(A^{-1}B^T\right)$

5. If A is an $n \times n$ matrix such that $\det(-A) = \det(A)$, what can you say about n?

6. Let A be a matrix such that $A^2 = A$. Prove that either A is singular or $\det(A) = 1$.

7. Prove that if A in an invertible matrix, then so is $A^T A$.

8. Prove that if P is an invertible matrix, then $\det(A) = \det(P^{-1}AP)$.

9. Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

This is called a **Vandermonde** determinant.

10. Show that if $a \neq 0$ $b \neq 0$ $c \neq 0$ then

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

11. Find all values of t such that $A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & t \\ 1 & t & -2 \end{bmatrix}$ is invertible.

12. For each of the given matrix A, find all real numbers λ such that $\det(\lambda I - A) = 0$. These values are called the **eigenvalues** of A.

a) $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ b) $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ d) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 6 & 0 & 2 \end{bmatrix}$

13. Let A be a 2×2 matrix. Show that if $\det(\lambda I - A) = 0$ where λ is a real number, then λ satisfies the equation

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$$

ANSWERS

1. a) $M_{21}=\begin{bmatrix}-1 & 2\\ 4 & 6\end{bmatrix}$ b) $m_{21}=-14$ c) $a_{21}=14$

   d) $M_{33}=\begin{bmatrix}3 & -1\\ 5 & 7\end{bmatrix}$ e) $m_{33}=26$ f) $a_{33}=26$

2. a) 24 b) -41 c) 151 d) –142 e) 147 f)10

   g) –108 h) 0 i) -100

3. a) $\det(A)=3$ b) $\det(B)=-10$ c) $\det(AB)=-30$

   d) $\det(A^{-1})=\dfrac{1}{3}$ e) $\det(A^T)=3$ f) $\det(A+B)=205$

4. a) $\det(AB)=-12$ b) $\det\left(A^{-1}\right)=\frac{1}{4}$ c) $\det(A^T)=4$

   d) $\det(4B)=-192$ e) $\det\left((AB)^T\right)=-12$ f) $\det\left((AB)^{-1}\right)=-\frac{1}{12}$

   g) $\det\left(B^6\right)=729$ h) $\det\left(A^{-1}B^T\right)=-\frac{3}{4}$

5. n is an even number

6.
$$\begin{aligned} A^2 &= A \\ \det\left(A^2\right) &= \det(A) \\ \det(A)\det(A)-\det(A) &= 0 \\ \det(A)\left(\det(A)-1\right) &= 0 \\ \det(A) &= -1 \text{ or } 0 \end{aligned}$$

   Hence either A is singular ($\det(A)=0$) or $\det(A)=1$.

7. If A is invertible, then $\det(A)\neq 0$ so

$$\det\left(AA^T\right)=\det(A)\det\left(A^T\right)=\det(A)\det(A)=\left[\det(A)\right]^2\neq 0$$

   hence A^TA is invertible.

8. If P is invertible, then $\det(P)\neq 0$ so

$$\det\left(P^{-1}AP\right)=\det\left(P^{-1}\right)\det(A)\det(P)=\frac{1}{\det(P)}\det(A)\det(P)=\det(A)$$

   hence $\det(A)=\det\left(P^{-1}AP\right)$.

9. $$\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} \overset{R_2 \to R_2 - R_1}{=} \begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 1 & c & c^2\end{vmatrix} \overset{R_3 \to R_3 - R_1}{=} \begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix}$$

$$= 1\cdot\left[(b-a)(c^2-a^2)-(c-a)(b^2-a^2)\right]$$
$$= (b-a)(c-a)(c+a)-(c-a)(b-a)(b+a)$$
$$= (b-a)(c-a)\left[(c+a)-(b+a)\right]$$
$$= (b-a)(c-a)(c-b)$$

10. $$\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix} = (1+a)\left[(1+b)(1+c)-1\right]-(1+c-1)+\left[1-(1+b)\right]$$

$$= (1+a)(b+c+bc)-c-b$$
$$= b+c+bc+ab+ac+abc-c-b$$
$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

11. $t \neq -1, \frac{7}{2}$

12. a) $\lambda = 2$, $\lambda = 3$ b) $\lambda = 0$, $\lambda = 7$ c) $\lambda = 2$, $\lambda = 3$, $\lambda = 1$
d) $\lambda = -1$, $\lambda = 0$, $\lambda = 4$

13. Let $A = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$. Thus $\operatorname{tr}(A) = a+d$ and $\det(A) = ad-bc$

Hence $\det(\lambda I - A) = \begin{vmatrix}\lambda - a & -b\\ -c & \lambda - d\end{vmatrix} = (\lambda - a)(\lambda - d) - bc$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$
$$= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$$