MATHEMATICS 201-105-RE
Linear Algebra
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III - Inverse and Elementary Matrices

1. Show that *B* is the inverse of *A*.

   a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

   b) $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

2. Find the inverse of the matrix (if it exists).

   a) $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ b) $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

   d) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

3. Use an inverse matrix to solve the given systems of linear equations.

   a) $\begin{aligned} -x + y &= 4 \\ -2x + y &= 0 \end{aligned}$ b) $\begin{aligned} -x + y &= -3 \\ -2x + y &= 5 \end{aligned}$ c) $\begin{aligned} -x + y &= 0 \\ -2x + y &= 0 \end{aligned}$

4. Use an inverse matrix to solve the given systems of linear equations.

   a) $\begin{aligned} 3x + 2y + 2z &= 0 \\ 2x + 2y + 2z &= 5 \\ -4x + 4y + 3z &= 2 \end{aligned}$ b) $\begin{aligned} 3x + 2y + 2z &= -1 \\ 2x + 2y + 2z &= 2 \\ -4x + 4y + 3z &= 0 \end{aligned}$ c) $\begin{aligned} 3x + 2y + 2z &= 0 \\ 2x + 2y + 2z &= 0 \\ -4x + 4y + 3z &= 0 \end{aligned}$

5. Using the following matrices,

   $A^{-1} = \begin{bmatrix} 2 & 5 \\ -7 & 6 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 7 & -3 \\ 2 & 0 \end{bmatrix}$

   find the following, using the properties of the inverse.

   a) $(AB)^{-1}$ b) $\left(A^T\right)^{-1}$ c) A^{-2} d) $(2A)^{-1}$

6. Find *A* such that

   a) $(2A)^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ b) $(I + 3A)^{-1} = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$

7. Prove that if C is an invertible matrix such that $CA = CB$, then $A = B$.

8. Determine whether the matrix is elementary. If it is, state the elementary row operation that was used to produce it.

a) $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ e) $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

9. Let A, B and C be given by

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 3 \\ -2 & -4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 & -4 & 5 \\ 1 & 3 & 3 \\ 2 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -4 & 4 \\ 1 & 3 & 3 \\ -2 & -4 & 5 \end{bmatrix}$$

a) Find an elementary matrix E such that $EA = B$.
b) Find an elementary matrix E such that $EA = C$.
c) Find an elementary matrix E such that $EB = A$.
d) Find an elementary matrix E such that $EC = A$.

10. Find the inverse of the given elementary matrix.

a) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $k \neq 0$

11. For each of the given matrices A, factor A^{-1} and A into a product of elementary matrices.

a) $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ c) $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

12. Prove that if B is invertible and $AB = BA$, then $AB^{-1} = B^{-1}A$.

13. Show that $(I - A)^{-1} = I + A + A^2 + A^3$ if $A^4 = 0$.

14. Let A, B and C be $n \times n$ matrices such that $AC = I$ and $CA = I$. Solve for C in terms of A and B.

a) $2A + 3C = 5B$
b) $(A + C)^2 - 2B = C(C - I)$
c) $(C + 3I)^2 = A + B$

Answers

1. a) $AB=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$ $BA=\begin{bmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$

b) $AB=\frac{1}{3}\begin{bmatrix}-2&2&3\\1&-1&0\\0&1&4\end{bmatrix}\begin{bmatrix}-4&-5&3\\-4&-8&3\\1&2&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

$BA=\frac{1}{3}\begin{bmatrix}-4&-5&3\\-4&-8&3\\1&2&0\end{bmatrix}\begin{bmatrix}-2&2&3\\1&-1&0\\0&1&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

2. a) $\begin{bmatrix}7&-2\\-3&1\end{bmatrix}$ b) $\begin{bmatrix}-19&-33\\-4&-7\end{bmatrix}$ c) Does not exist

d) $\begin{bmatrix}1&1&-1\\-3&2&-1\\3&-3&2\end{bmatrix}$ e) $\begin{bmatrix}-\frac{3}{2}&\frac{3}{2}&1\\\frac{9}{2}&-\frac{7}{2}&-3\\-1&1&1\end{bmatrix}$ f) $\begin{bmatrix}1&0&0\\-\frac{3}{4}&\frac{1}{4}&0\\\frac{7}{20}&-\frac{1}{4}&\frac{1}{5}\end{bmatrix}$

3. a) $x=4,\ y=8$ b) $x=-8,\ y=-11$ c) $x=0,\ y=0$

4. a) $x=-5,\ y=-\frac{81}{2},\ z=48$ b) $x=-3,\ y=-24,\ z=28$ c) $x=0,\ y=0,\ z=0$

5. a) $\begin{bmatrix}35&17\\4&10\end{bmatrix}$ b) $\begin{bmatrix}2&-7\\5&6\end{bmatrix}$ c) $\begin{bmatrix}-31&40\\-56&1\end{bmatrix}$ d) $\begin{bmatrix}1&\frac{5}{2}\\-\frac{7}{2}&3\end{bmatrix}$

6. a) $\begin{bmatrix}\frac{5}{26}&\frac{1}{26}\\-\frac{3}{26}&\frac{1}{13}\end{bmatrix}$ b) $\begin{bmatrix}-\frac{4}{15}&-\frac{2}{15}\\\frac{1}{30}&-\frac{7}{30}\end{bmatrix}$

7. We have $CA=CB$. Since C is invertible, then multiplying by C^{-1}, we have
$$C^{-1}CA=C^{-1}CB$$
$$IA=IB$$
$$A=B$$

8. a) Yes: $R_2\to\sqrt{2}R_2$ b) No c) No d) Yes: $R_2\to R_2+2R_3$ e) No f) No

9. a) $E=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ b) $E=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$ c) $E=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ d) $E=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$

10. a) $\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$ b) $\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$ c) $\begin{bmatrix}1&0&0\\0&\frac{1}{k}&0\\0&0&1\end{bmatrix}$

11. a) $A^{-1}=\begin{bmatrix}1&0\\-4&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}$ $A=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\4&1\end{bmatrix}$

b) $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

c)
$$A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d)
$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

12. We have $AB = BA$. Since B is invertible, multiplying both sides, on the left and on the right, by B^{-1} we obtain

$$B^{-1}ABB^{-1} = B^{-1}BAB^{-1}$$
$$B^{-1}AI = IAB^{-1}$$
$$B^{-1}A = AB^{-1}$$

which is equivalent to $AB^{-1} = B^{-1}A$.

13. Let us show that the inverse of $I - A$ is $I + A + A^2 + A^3$ if $A^4 = 0$.

$$\begin{aligned}(I - A)(I + A + A^2 + A^3) &= II + IA + IA^2 + IA^3 - AI - AA - AA^2 - AA^3 \\ &= I + A + A^2 + A^3 - A - A^2 - A^3 - A^4 \\ &= I \qquad (\text{ Since } A^4 = 0 \text{ })\end{aligned}$$

14. a) $C = \frac{5}{3}B - \frac{2}{3}A$ b) $C = 2B - A^2 - 2I$ c) $C = A^2 + AB - 6I - 9A$