MATHEMATICS 201-105-RE
Linear Algebra
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I - Matrices

1. Let $A=\begin{bmatrix} 2 & -3 & 5 \\ 1 & \sqrt{2} & 7 \\ \pi & 3 & 0 \end{bmatrix}$. Find a_{23}, a_{12} and a_{31}.

2. Consider the following matrices.

$$A=\begin{bmatrix} 2 & -3 & 5 \\ 0 & 7 & 7 \\ 0 & 0 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \quad C=\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad D=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F=\begin{bmatrix} 3 & -2 & 4 \\ -2 & 0 & 3 \\ 4 & 3 & 5 \end{bmatrix} \quad G=\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \quad H=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Determine which of these matrices (if any) are
   a) lower triangular
   b) upper triangular
   c) square
   d) diagonal
   e) column
   f) row
   g) symmetric

3. Consider the following matrices.

$$A=\begin{bmatrix} -1 & -3 \\ 4 & 2 \\ 2 & -1 \end{bmatrix} \quad B=\begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix} \quad C=\begin{bmatrix} 3 & 4 & 2 \\ -2 & 4 & -3 \end{bmatrix} \quad D=\begin{bmatrix} 5 & -1 & 2 \\ 2 & 0 & -4 \\ 1 & 3 & -2 \end{bmatrix} \quad E=\begin{bmatrix} 2 & -1 & 0 \\ 5 & -2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

Compute the following (where possible).

a) $D+E$	b) $D-E$	c) $3C$	d) $-2A$
e) $2D+4E$	f) $3B-C$	g) $-2(2E-4D)$	h) $E-E$
i) $\text{tr}(D)$	j) $\text{tr}(C)$	k) $\text{tr}(2D-5E)$	l) $\text{tr}(5E)$

4. Using the matrices defined in 3, compute the following.

a) $2B^T$	b) C^T-A	c) $2E^T-3D$	d) $\left(2A^T-3C\right)^T$
e) AB	f) BA	g) AC	h CA
i) $C(DE)$	j) C^TB	k) BD	l) $\text{tr}(B-A^TC^T)$
m) $\left(DE\right)^T-D^TE^T$	n) AA^T	o) D^2	p) D^3

5. Consider the matrix $A=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Evaluate A^2, A^3, A^n.

6. Find x and y such that

$$\begin{bmatrix} x & 3 \\ -6 & y \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix}=\begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

7. Solve for A.

a) $\begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}-4A=\begin{bmatrix} 3 & 5 \\ -2 & 7 \end{bmatrix}$

b) $2A-3\begin{bmatrix} 3 & -2 \\ 4 & 4 \end{bmatrix}=\begin{bmatrix} -2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

8. Prove that if A is a square matrix, then $A+A^T$ is symmetric.

9. Let A be a symmetric matrix.

a) Show that A^2 is symmetric.

b) Show that $3A^2+2A-3I$ is symmetric.

10. Prove that if $A^T A = A$, then

a) A is symmetric

b) $A = A^2$

11. Let A and B be two matrices defined by

$$A=\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \qquad \text{and } B = I - A$$

a) Prove that $A^2 = A$ and $B^2 = B$

b) Evaluate AB and BA.

12. A matrix B is said to be the ***square root*** of a matrix A if $B^2 = A/$

a) Find two square roots of $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

b) Find all square roots of $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$.

13. A furniture shop makes non-painted desks, chairs and tables out of wood. The time it takes to make an item is given in the following chart.

	Desk	Chair	Table
Sawing	3	3	3
Assembling	2	1	2
Sanding	2	1	2

a) The shop has an order for 25 desks, 32 chairs and 16 tables. Determine the time needed to complete the order of each of the workshops.
b) Knowing that the salary for each of the workers for sawing is $12.75 an hour, for assembly $9.05 and for sanding $10.50, determine the production costs in salary for the order.
c) Determine the cost for producing one item of each kind.

14. A clerk in a grocery store prepares coffee mixes from three different kinds of coffee: Kenyan, Peruvian and Columbian. The quantities necessary, in kilograms, to make one kilogram of each mix is given in the following chart.

	M_1	M_2	M_3
Kenyan	.3	.5	.4
Peruvian	.5	.2	.2
Columbian	.2	.3	.4

a) Knowing that the store sells every week 30 kg of the first mix, 20 kg of the second mix and 50 kg of the third mix, how many kilograms of each kind of coffee must the clerk order every week?
b) The store buys the coffee at a price of $5.85 the kilogram for the Kenyan coffee, $5.75 for the Peruvian and $4.25 for the Columbian. Find how much it costs the store to make one kilogram of each mix.
c) Knowing that the store makes a 120% profit, find at what price each mix must be sold.

Answers

1. 7, -3, π
2. a) B, C, H b) A, H c) A, B, C, F, H d) H
   e) G f) None g) F, H
3. a) $\begin{bmatrix} 7 & -2 & 2 \\ 7 & -2 & 1 \\ 3 & 4 & -1 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 0 & 2 \\ -3 & 2 & -9 \\ -1 & 2 & -3 \end{bmatrix}$ c) $\begin{bmatrix} 9 & 12 & 6 \\ -6 & 12 & -9 \end{bmatrix}$

   d) $\begin{bmatrix} 2 & 6 \\ -8 & -4 \\ -4 & 2 \end{bmatrix}$ e) $\begin{bmatrix} 18 & -6 & 4 \\ 24 & -8 & 12 \\ 10 & 10 & 0 \end{bmatrix}$ f) Undefined g) $\begin{bmatrix} 32 & -4 & 16 \\ -4 & 8 & -52 \\ 0 & 20 & -20 \end{bmatrix}$

   h) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ i) 3 j) Undefined k) 1 l) 5
4. a) $\begin{bmatrix} 4 & 8 \\ -6 & -4 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 1 \\ 0 & 2 \\ 0 & -2 \end{bmatrix}$ c) $\begin{bmatrix} -11 & 13 & -2 \\ -8 & -4 & 14 \\ -3 & 1 & 8 \end{bmatrix}$ d) $\begin{bmatrix} -11 & 0 \\ -4 & -8 \\ -2 & 7 \end{bmatrix}$

   e) $\begin{bmatrix} -14 & 9 \\ 16 & -16 \\ 0 & -4 \end{bmatrix}$ f) Undefined g) $\begin{bmatrix} 3 & -16 & 7 \\ 8 & 24 & 2 \\ 8 & 4 & 7 \end{bmatrix}$ h) $\begin{bmatrix} 17 & -3 \\ 12 & 17 \end{bmatrix}$

   i) $\begin{bmatrix} 37 & -45 & 1 \\ -73 & 5 & -49 \end{bmatrix}$ j) $\begin{bmatrix} -2 & -5 \\ 24 & -20 \\ -8 & 0 \end{bmatrix}$ k) Undefined l) –34

   m) $\begin{bmatrix} 1 & -30 & 0 \\ 1 & -16 & -10 \\ -11 & -12 & 15 \end{bmatrix}$ n) $\begin{bmatrix} 10 & -10 & 1 \\ -10 & 20 & 6 \\ 1 & 6 & 5 \end{bmatrix}$ o) $\begin{bmatrix} 25 & 1 & 10 \\ 6 & -14 & 12 \\ 9 & -7 & -6 \end{bmatrix}$

   p) $\begin{bmatrix} 137 & 5 & 26 \\ 14 & 30 & 44 \\ 25 & -27 & 58 \end{bmatrix}$
5. $A^2 = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$ $A^3 = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$ $A^n = \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix}$
6. $x = -\frac{7}{2}$ $y = \frac{9}{4}$
7. a) $A = \begin{bmatrix} \frac{-1}{4} & \frac{-7}{4} \\ \frac{5}{4} & \frac{-3}{2} \end{bmatrix}$ b) $A = \begin{bmatrix} \frac{5}{2} & \frac{-1}{2} \\ 11 & 8 \end{bmatrix}$

8. $\left(A+A^T\right)^T = A^T + \left(A^T\right)^T = A^T + A = A + A^T$

9. a) $\left(A^2\right)^T = \left(AA\right)^T = A^T A^T = AA = A^2$ (since A is symmetric, $A^T = A$)

b) $\left(3A^2 + 2A - 3I\right)^T = 3\left(A^2\right)^T + 2A^T - 3I^T$

$= 3A^T A^T + 2A^T - 3I$

$= 3A^2 + 2A - 3I$ (since A is symmetric, $A^T = A$)

10. a) $A = A^T A$

$A^T = \left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A = A$

b) $A = A^T A$

$= AA = A^2$ since by (a) A is symmetric

11. a) $A^2 = \begin{bmatrix} \cos^4\theta + \sin^2\theta\cos^2\theta & \cos^3\theta\sin\theta + \sin^3\theta\cos\theta \\ \cos^3\theta\sin\theta + \sin^3\theta\cos\theta & \sin^2\theta\cos^2\theta + \sin^4\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta\left(\cos^2\theta + \sin^2\theta\right) & \cos\theta\sin\theta\left(\cos^2\theta + \sin^2\theta\right) \\ \cos\theta\sin\theta\left(\cos^2\theta + \sin^2\theta\right) & \sin^2\theta\left(\cos^2\theta + \sin^2\theta\right) \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} = A$

$B^2 = \left(I - A\right)^2 = I^2 - IA - AI + A^2 = I - A - A + A = I - A = B$ (since $A^2 = A$)

b) $AB = A\left(I - A\right) = AI - A^2 = A - A = 0$

$BA = \left(I - A\right)A = IA - A^2 = A - A = 0$

12. a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ b) $\begin{bmatrix} \pm 2 & 0 \\ 0 & \pm 3 \end{bmatrix}$ (4 different matrices).

13. a) 219 hours of sawing, 114 hours of assembling and 114 hours of sanding.
    b) \$ 5020.95
    c) \$77.35 for a desk, \$57.80 for a chair and \$77.35 for a table.

14. a) 39 kg of Kenyan, 29 kg of African and 32 kg of Columbian
    b) \$5.48 for a kilogram of M_1, \$5.35 for a kilogram of M_2 and \$5.19 for a kilogram of M_3.
    c) \$12.06 for a kilogram of M_1, \$11.77 for a kilogram of M_2 and \$11.42 for a kilogram of M_3.