

MATHEMATICS 201-105-RE

Linear Algebra

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Winter 2006

Assignment #4 SOLUTIONS

This assignment is due **Friday April 21**. Complete solutions are expected.

Question 1 (8 points)

Prove that the set $W = \{(t, 3) : t \in \mathbb{R}\}$ with the following definitions for vector sum and scalar multiplication is a vector space. (Verify all 10 axioms!)

$$(u_1, 3) \oplus (v_1, 3) = (u_1 + v_1, 3)$$

$$k \odot (u_1, 3) = (ku_1, 3)$$

Let $\vec{u} = (u_1, 3)$, $\vec{v} = (v_1, 3)$ and $\vec{w} = (w_1, 3)$ be in W .

- $\vec{u} \oplus \vec{v} = (u_1, 3) \oplus (v_1, 3) = (u_1 + v_1, 3) \in W$ since $u_1 + v_1 \in \mathbb{R}$
- $\vec{u} \oplus \vec{v} = (u_1 + v_1, 3) = (v_1 + u_1, 3) = \vec{v} \oplus \vec{u}$
- $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (u_1, 3) \oplus (v_1 + w_1, 3)$
 $= (u_1 + (v_1 + w_1), 3)$
 $= ((u_1 + v_1) + w_1, 3) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$
- $\vec{0} = 0 \odot \vec{u} = 0 \odot (u_1, 3) = (0 \cdot u_1, 3) = (0, 3)$
 $\vec{u} \oplus \vec{0} = (u_1, 3) \oplus (0, 3) = (u_1 + 0, 3) = (u_1, 3) = \vec{u}$
- $-\vec{u} = (-1) \odot \vec{u} = (-1) \odot (u_1, 3) = (-u_1, 3) = (-u_1, 3)$
 $\vec{u} \oplus (-\vec{u}) = (u_1, 3) \oplus (-u_1, 3) = (u_1 - u_1, 3) = (0, 3) = \vec{0}$
- $k \odot \vec{u} = (ku_1, 3) \in W$ since $ku_1 \in \mathbb{R}$
- $k \odot (\vec{u} \oplus \vec{v}) = k \odot (u_1 + v_1, 3) = (k(u_1 + v_1), 3)$
 $= (ku_1 + kv_1, 3) = (ku_1, 3) \oplus (kv_1, 3) = (k \odot \vec{u}) \oplus (k \odot \vec{v})$
- $(k+l) \odot \vec{u} = ((k+l)u_1, 3) = (ku_1 + lu_1, 3)$
 $= (ku_1, 3) \oplus (lu_1, 3) = (k \odot \vec{u}) \oplus (l \odot \vec{u})$
- $k \odot (l \odot \vec{u}) = k \odot (lu_1, 3) = (k(lu_1), 3) = (kl u_1, 3) = (kl) \odot \vec{u}$
- $1 \odot \vec{u} = 1 \odot (u_1, 3) = (1 \cdot u_1, 3) = (u_1, 3) = \vec{u}$

Question 2 (4 points)

Does the set $S = \{x^2 + x + 1, x^2 - x + 3, 2x^2 - x + 5\}$ span P_2 ?

Let $p(x) = ax^2 + bx + c \in P_2$.

$$c_1(x^2 + x + 1) + c_2(x^2 - x + 3) + c_3(2x^2 - x + 5) = ax^2 + bx + c$$

$$\begin{array}{l} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & -1 & -1 & b \\ 1 & 3 & 5 & c \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -2 & -3 & b-a \\ 0 & 2 & 3 & c-a \end{array} \right] \end{array}$$

$$\begin{array}{l} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -2 & -3 & b-a \\ 0 & 0 & 0 & b+c-2a \end{array} \right] \begin{array}{l} \\ \\ R_3 \rightarrow R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -2 & -3 & b-a \\ 0 & 0 & 0 & b+c-2a \end{array} \right] \end{array}$$

There is no solution if $b+c-2a \neq 0$, so S does not span P_2 .

Question 3 (7 points)

Consider the set of symmetric 2×2 matrices $S_2 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix}; a, b, c \in \mathbb{R} \right\}$.

a) Show that S_2 is a subspace of $M_{2,2}$, the set of 2×2 matrices.

S_2 is nonempty since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S_2$.

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $B = \begin{bmatrix} r & s \\ s & t \end{bmatrix} \in S_2$

$$1. A+B = \begin{bmatrix} a+r & b+s \\ b+s & c+t \end{bmatrix} \in S_2 \quad \text{since } A+B \text{ is still symmetrical}$$

$$2. kA = \begin{bmatrix} ka & kb \\ kb & kc \end{bmatrix} \in S_2 \quad \text{since } kA \text{ is still symmetrical}$$

Thus S_2 is a subspace of $M_{2,2}$

b) Find a basis for S_2 .

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in S_2$

$$\text{Then } \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus if $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, then $S_2 = \text{span}(B)$.

For linear independence,

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{array}$$

so B is linearly independent

Ergo B is a basis for S_2

c) What is the dimension of S_2 ? $\dim(S_2) = n(B) = 3$

Question 4 (5 points)

For what values of t is the set $S = \{(2, -3, 3), (3, -2, t), (-1, t, 4)\}$ linearly independent?

$$c_1(2, -3, 3) + c_2(3, -2, t) + c_3(-1, t, 4) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ -3 & -2 & t & 0 \\ 3 & t & 4 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 + 3R_1 \\ R_3 \rightarrow 2R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 5 & 2t-3 & 0 \\ 0 & 2t-9 & 11 & 0 \end{array} \right]$$

$$\underline{R_3 \rightarrow 5R_3 - (2t-9)R_1} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 5 & 2t-3 & 0 \\ 0 & 0 & -4t^2 + 24t + 28 & 0 \end{array} \right]$$

$$\begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{5}R_2 \\ R_3 \rightarrow \frac{1}{-4t^2+24t+28}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{2}{5}t - \frac{3}{5} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{array}$$



Illegal if $-4t^2 + 24t + 28 = 0$

$$t^2 - 6t - 7 = 0$$

$$(t-7)(t+1) = 0$$

$$t = -1, 7$$

If $t \neq -1, 7$ then S is linearly independent.

If $t = -1$ then

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{5}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = t \\ c_2 = t \\ c_1 = -t \end{array}$$

So S is linearly dependent

If $t = 7$ then

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 5 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{5}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{11}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = t \\ c_2 = \frac{-11}{5}t \\ c_1 = \frac{-26}{10}t \end{array}$$

So S is linearly dependent

Thus S is linearly independent if $t \neq -1, 7$.

Question 5 (5 points)Do the following sets B form a basis for V ?

$$\text{a) } B = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right\} \quad V = M_{2,2}$$

No since $n(B) = 5 \neq \dim(M_{2,2}) = 4$

$$\text{b) } B = \{(2, -1, 3), (4, 1, 5), (-2, -5, -1)\} \quad V = \mathbb{R}^3$$

Verify for linear independence

$$c_1(2, -1, 3) + c_2(4, 1, 5) + c_3(-2, -5, -1) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ -1 & 1 & -5 & 0 \\ 3 & 5 & -1 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow 2R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 0 & 6 & -12 & 0 \\ 0 & -2 & 4 & 0 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow 3R_3 + R_2 \\ R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{6}R_2 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$c_3 = t$$

$$c_2 = 2t$$

$$c_1 = -3t$$

Thus B is linearly dependentErgo, B is not a basis for \mathbb{R}^3

Question 6 (13 points)

Consider the subset $W = \{(t, 3t + 2s, s) \mid s, t \in \mathbb{R}\}$ of \mathbb{R}^3 .

- a) Show that W is a subspace of \mathbb{R}^3 .

W is nonempty since $\vec{0} = (0, 0, 0) \in W$

Let $\vec{u} = (a, 3a + 2b, b)$ and $\vec{v} = (c, 3c + 2d, d) \in W$.

$$\begin{aligned} 1. \quad \vec{u} + \vec{v} &= (a + c, 3a + 2b + 3c + 2d, b + d) \\ &= (a + c, 3(a + c) + 2(b + d), b + d) \in W \end{aligned}$$

$$2. \quad k\vec{u} = (ka, k(3a + 2b), kb) = (ka, 3(ka) + 2(kb), kb) \in W$$

- b) Find a basis for W .

If $\vec{u} = (a, 3a + 2b, b) \in W$

then $(a, 3a + 2b, b) = a(1, 3, 0) + b(0, 2, 1)$

Thus if $B = \{(1, 3, 0), (0, 2, 1)\}$, then $W = \text{span}(B)$, and since the two vectors are not multiples of each other, then B is linearly independent, so it forms a basis for W .

- c) Find the dimension of W . $\dim(W) = n(B) = 2$

- d) Give a geometrical description of W .

$$\vec{n} = (1, 3, 0) \times (0, 2, 1) = \begin{vmatrix} i & j & k \\ 1 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix} = (3, -1, 2)$$

Thus W is the plane $\pi: 3x - y + 2z = 0$

- e) Find the coordinate vector for $\vec{v} = (4, 22, 5) \in W$ relative to the basis found in (b).

$$c_1(1, 3, 0) + c_2(0, 2, 1) = (4, 22, 5)$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 3 & 2 & 22 \\ 0 & 1 & 5 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 2 & 10 \\ 0 & 1 & 5 \end{array} \right] \xrightarrow{R_3 \rightarrow 2R_3 - R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 2 & 10 \\ 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array} \right] \quad \begin{aligned} c_2 &= 5 \\ c_1 &= 4 \end{aligned}$$

Thus $(\vec{v})_B = (4, 5)$

Question 7 (8 points)

Consider the set $A = \{(-1, 2, 3), (3, -2, 1), (11, -10, -3), (1, 0, 2)\}$

a) Find a basis for $\text{span}(A)$.

$$c_1(-1, 2, 3) + c_2(3, -2, 1) + c_3(11, -10, -3) + c_4(1, 0, 2) = (0, 0, 0)$$

$$\left[\begin{array}{cccc|c} -1 & 3 & 11 & 1 & 0 \\ 2 & -2 & -10 & 0 & 0 \\ 3 & 1 & -3 & 2 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 + 3R_1 \end{array} \left[\begin{array}{cccc|c} -1 & 3 & 11 & 1 & 0 \\ 0 & 4 & 12 & 2 & 0 \\ 0 & 10 & 30 & 5 & 0 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow 2R_3 - 5R_1 \\ R_1 \rightarrow -R_1 \\ R_2 \rightarrow \frac{1}{4}R_2 \end{array} \left[\begin{array}{cccc|c} -1 & 3 & 11 & 1 & 0 \\ 0 & 4 & 12 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc|c} 1 & -3 & -11 & -1 & 0 \\ 0 & 1 & 3 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$c_4 = t$$

$$c_3 = s$$

$$c_2 = -3s - \frac{1}{2}t$$

$$c_1 = 3(-3s - \frac{1}{2}t) + 11s + t = 2s - \frac{1}{2}t$$

If $s = 1$ and $t = 0$

$$2(-1, 2, 3) - 3(3, -2, 1) + 1(11, -10, -3) + 0(1, 0, 2) = (0, 0, 0)$$

$$(11, -10, -3) = -2(-1, 2, 3) + 3(3, -2, 1)$$

If $s = 0$ and $t = 1$

$$\frac{-1}{2}(-1, 2, 3) - \frac{1}{2}(3, -2, 1) + 0(11, -10, -3) + 1(1, 0, 2) = (0, 0, 0)$$

$$(1, 0, 2) = \frac{1}{2}(-1, 2, 3) + \frac{1}{2}(3, -2, 1)$$

By the +/- theorem, if $B = \{(-1, 2, 3), (3, -2, 1)\}$, then $\text{span}(B) = \text{span}(A)$.

Since the two vectors are not multiples of each other, they are linearly independent, thus B is a basis for $\text{span}(A)$.

b) Find the dimension of $\text{span}(A)$. $\dim(\text{span}(A)) = n(B) = 2$

c) Give a geometrical interpretation for $\text{span}(A)$.

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = (8, 10, -4)$$

Thus $\text{span}(A)$ is the plane $\pi: 4x + 5y - 2z = 0$