MATHEMATICS 201-105-RE
Linear Algebra
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Assignment #3
SOLUTIONS

This assignment is due ***Friday March 31.*** Complete solutions are expected. For questions 3 and 6, a print out of your Maple is expected, where each question is clearly indicated.

Question 1 (10 points)

Consider the points $A(-3,-4)$ and $B(5,-2)$ and $C(-5,1)$.

a) Find the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\cos\theta = \frac{\overrightarrow{AB}\bullet\overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|} = \frac{(8,2)\bullet(-2,5)}{\|(8,2)\|\|(-2,5)\|} = \frac{-6}{\sqrt{68}\sqrt{29}}$$

$\theta \approx 97.8°$

b) Find the orthogonal projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.

$$\text{proj}_{\overrightarrow{AC}}\overrightarrow{AB} = \frac{\overrightarrow{AB}\bullet\overrightarrow{AC}}{\overrightarrow{AC}\bullet\overrightarrow{AC}}\overrightarrow{AC} = \frac{(8,2)\bullet(-2,5)}{(-2,5)\bullet(-2,5)}(-2,5) = \frac{-6}{29}(-2,5) = \left(\frac{12}{29},\frac{-30}{29}\right)$$

c) Find the area of the triangle ABC.

$$\text{Area} = \tfrac{1}{2}\left\|\overrightarrow{AB_0}\times\overrightarrow{AC_0}\right\| = \tfrac{1}{2}\left\|(8,2,0)\times(-2,5,0)\right\| = \tfrac{1}{2}\left\|(0,0,44)\right\| = 22$$

d) Find the equation of the line L (in parametric form and in general form) parallel to $L_1: 3x-4y=12$ and passing through the point C.

$\vec{n} = \vec{n}_1 = (3,-4)$ $\qquad\qquad$ $\vec{u} = (4,3)$

Parametric form: $L:\begin{cases} x = -5+4t \\ y = 1+3t \end{cases}$

General form: $3x-4y = 3(-5)-4(1)$

$L: 3x-4y = -19$

e) Find the distance between the line $L_1: 3x-4y=12$ and the point B.

$R(4,0)$

$\overrightarrow{BR} = (-1,2)$

$$d = \frac{\left|\overrightarrow{BR}\bullet\vec{n}\right|}{\|\vec{n}\|} = \frac{|(-1,2)\bullet(3,-4)|}{\|(3,-4)\|} = \frac{11}{5}$$

Question 2 (10 points)

Consider the lines $L_1 : \frac{x+1}{2} = \frac{y-3}{4} = \frac{3-z}{5}$ and $L_2 : \begin{cases} x = 2+3t \\ y = 5-4t \\ z = 5 \end{cases}$.

a) Find the relationship between L_1 and L_2. (That is, are they parallel and distinct, parallel and equivalent, intersecting or skew?)

$L_1 : (x,y,z) = (-1,3,3) + t(2,4,-5) \qquad u_1 = (2,4,-5)$

$L_2 : (x,y,z) = (2,5,5) + t(3,-4,0) \qquad u_2 = (3,-4,0)$

Since $u_1 \not\parallel u_2$ then $L_1 \not\parallel L_2$

$$(-1,3,3) + t(2,4,-5) = (2,5,5) + s(3,-4,0)$$

$$t(2,4,-5) + s(-3,4,0) = (3,2,2)$$

$$\left[\begin{array}{cc|c} 2 & -3 & 3 \\ 4 & 4 & 2 \\ -5 & 0 & 2 \end{array}\right] \xrightarrow[R_3 \to 2R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & -3 & 3 \\ 0 & 10 & -4 \\ 0 & 15 & -11 \end{array}\right]$$

$$\xrightarrow{R_3 \to 2R_3 - 3R_2} \left[\begin{array}{cc|c} 2 & -3 & 3 \\ 0 & 10 & -4 \\ 0 & 0 & -10 \end{array}\right] \xrightarrow[R_3 \to \frac{-1}{10} R_3]{R_1 \to \frac{1}{2} R_1,\; R_2 \to \frac{1}{10} R_2} \left[\begin{array}{cc|c} 1 & \frac{-3}{2} & \frac{3}{2} \\ 0 & 1 & \frac{-2}{5} \\ 0 & 0 & 1 \end{array}\right]$$

Since there is no solution, then L_1 and L_2 are skew.

b) Find the equation of the line L_3 (in symmetric form) parallel to L_1 and passing through the point $P(-1,13,-16)$.

$$L_3 : \frac{x+1}{2} = \frac{y-13}{4} = \frac{z+16}{-5}$$

c) Find the distance between L_1 and $P(-1,13,-16)$.

$R(-1,3,3)$

$\overrightarrow{PR} = (0,-10,19)$

$$d = \frac{\left\|\overrightarrow{PR} \times \vec{u}\right\|}{\|\vec{u}\|} = \frac{1}{\sqrt{45}} \left\| \begin{vmatrix} i & j & k \\ 0 & -10 & 19 \\ 2 & 4 & -5 \end{vmatrix} \right\| = \frac{1}{3\sqrt{5}} \|(-26,38,20)\| = \frac{\sqrt{5}}{15}\sqrt{2520} = 2\sqrt{14}$$

d) Find the distance between L_1 and L_2.

$$u_1 \times u_2 = \begin{vmatrix} i & j & k \\ 2 & 4 & -5 \\ 3 & -4 & 0 \end{vmatrix} = (-20,-15,-20)$$

$P_1(-1,3,3)$

$P_2(2,5,5)$

$\overrightarrow{P_1P_2} = (3,2,2)$

$$d = \frac{\left|\overrightarrow{P_1P_2} \bullet (\vec{u}_1 \times \vec{u}_2)\right|}{\|\vec{u}_1 \times \vec{u}_2\|} = \frac{130}{\sqrt{1025}} = \frac{26\sqrt{41}}{41}$$

e) Find the point Q on L_1 that is closest to the point $P(-1,13,-16)$.

$\vec{u}=(2,4,-5)$

$R(-1,3,3)$

$\overrightarrow{RP}=(0,10,-19)$

$$\overrightarrow{RQ}=\text{proj}_{\vec{u}}\,\overrightarrow{RP}=\frac{\overrightarrow{RP}\bullet\vec{u}}{\vec{u}\bullet\vec{u}}\vec{u}$$

$$(x+1,y-3,z-3)=\frac{135}{45}(2,4,-5)$$

$$(x,y,z)=3(2,4,-5)+(-1,3,3)=(5,15,-12)$$

Thus $Q(5,15,-12)$

Question 3 (6 points)

Consider the points $A(3,2,1)$, $B(-2,4,-6)$ and $C(4,0,-5)$. Using Maple,

a) find the equation of the line l passing through the points A and B;
b) plot the line l found in (a) along with the direction vector for the line;
c) find the distance between the point C and the line l found in (a).

Question 4 (8 points)

Consider the plane $\pi_1 : 3x - 2y + 4z = 6$.

a) Find the equation of the line *l* (in symmetric form) perpendicular to π_1 and passing through the point $P(-2,5,-9)$.

$$l : \frac{x+2}{3} = \frac{5-y}{2} = \frac{z+9}{4}$$

b) Find the distance between the point $P(-2,5,-9)$ and the plane π_1.

$\vec{n} = (3,-2,4)$

$R(2,0,0)$

$\overrightarrow{PR} = (4,-5,9)$

$$d = \frac{\left|\overrightarrow{PR} \bullet \vec{n}\right|}{\|\vec{n}\|} = \frac{58}{\sqrt{29}} = 2\sqrt{29}$$

c) Find the point Q on π_1 that is closest to the point $P(-2,5,-9)$.

$\vec{n} = (3,-2,4)$

$R(2,0,0)$

$\overrightarrow{PR} = (4,-5,9)$

$$\overrightarrow{PQ} = \text{proj}_{\vec{n}}\, \overrightarrow{PR} = \frac{\overrightarrow{PR} \bullet \vec{n}}{\vec{n} \bullet \vec{n}} \vec{n}$$

$$(x+2, y-5, z+9) = \frac{58}{29}(3,-2,4)$$

$$(x,y,z) = 2(3,-2,4) + (-2,5,-9) = (4,1,-1)$$

Thus $Q(4,1,-1)$

d) Find the equation of the plane (in general form) perpendicular to π_1 and containing the line $l_1 : \frac{x-1}{2} = \frac{y}{2} = \frac{1-z}{3}$.

$$\vec{n} = \vec{n}_1 \times \vec{u} = \begin{vmatrix} i & j & k \\ 3 & -2 & 4 \\ 2 & 2 & -3 \end{vmatrix} = (-2,17,10)$$

$$-2x + 17y + 10z = -2(1) + 17(0) + 10(1) = 8$$

Thus the plane is $2x - 17y - 10z = -8$

Question 5 (2 points)

Find the equation of the plane (in general form), if possible, containing the lines
$L_1 : \frac{x+1}{2} = \frac{y-3}{4} = 2z$ and $L_2 : \frac{x+1}{4} = \frac{x-4}{8} = z-3$.

$\vec{u}_1 = \left(2,4,\frac{1}{2}\right)$ $\therefore L_1 \parallel L_2$

$\vec{u}_2 = (4,8,1)$ $P_1(-1,3,0) \in L_1$ $P_1(-1,3,0) \in L_2$?

$\frac{-1+1}{4} = \frac{3-4}{8} = 0-3$ $P_1 \notin L_2$

$0 \neq \frac{-1}{8} \neq -3$

Thus L_1 and L_2 are parallel and distinct.

$P_2(-1,4,3)$ $\overrightarrow{P_1P_2} = (0,1,3)$

Thus $\vec{n} = \overrightarrow{P_1P_2} \times \vec{u} = \begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 4 & 8 & 1 \end{vmatrix} = (-23,12,-4)$.

$-23x+12y-4z = -23(-1)+12(3)-4(0) = 59$

Ergo, the equation of the plane is $23x-12y+4z=-59$

Question 6 (10 points)

Consider the four points $A(-1,3,4)$, $B(3,3,0)$, $C(2,-1,3)$ and $E(1,3,1)$. Using Maple,

a) find the equation of the plane π (in general form) passing through the points *A*, *B* and *C*;
b) find the equation of the line *l* (in vector form) passing through *E* and perpendicular to the plane π;
c) find the distance between the point *E* and the plane π;
d) plot the line *l*, the plane π and the normal vector;
e) find the volume of the tetrahedron *ABCE*.

Question 7 (4 points)

Consider the planes $\pi_1 : x+y-2z=5$ and $\pi_2 : 2x+y+3z=4$.

a) Find the intersection of the planes π_1 and π_2 (expressed in vector form).

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 2 & 1 & 3 & 4 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & -1 & 7 & -6 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & 1 & -7 & 6 \end{array}\right]$$

$z = t$

$y = 6+7t$ Ergo, $\pi_1 \cap \pi_2 : (x,y,z) = (-1,6,0)+t(-5,7,1)$

$z = 5+2t-(6+7t) = -1-5t$

b) Find the angle between the two planes.

$$\cos\theta = \frac{|\vec{n}_1 \bullet \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|} = \frac{|(1,1,-2)\bullet(2,1,3)|}{\|(1,1,-2)\|\|(2,1,3)\|} = \frac{|-3|}{\sqrt{6}\sqrt{14}} \qquad \theta \approx 70.9^\circ$$