MATHEMATICS 201-105-RE
Linear Algebra
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Winter 2005

Assignment #2

SOLUTIONS

This assignment is due ***Friday February 24.*** Complete solutions are expected.

Question 1 (4 points)

Find the determinant of $A=\begin{bmatrix}1&0&2&0\\0&3&0&4\\5&0&-1&6\\0&7&8&0\end{bmatrix}$.

$$\begin{aligned}\det(A)&=1\begin{vmatrix}3&0&4\\0&-1&6\\7&8&0\end{vmatrix}-0+2\begin{vmatrix}0&3&4\\5&0&6\\0&7&0\end{vmatrix}-0\\&=\left(3\begin{vmatrix}-1&6\\8&0\end{vmatrix}-0+4\begin{vmatrix}0&-1\\7&8\end{vmatrix}\right)+2\left(0-5\begin{vmatrix}3&4\\7&0\end{vmatrix}+0\right)\\&=3(-48)+4(7)-10(-28)\\&=164\end{aligned}$$

Question 2 (11 points)

Consider the following system of linear equations

$$\begin{aligned} x + 2y + 3z &= 13 \\ 3x \quad\quad - 2z &= 2 \\ x - 4y \quad\quad &= 16 \end{aligned}$$

a) Solve the system using <u>Cramer's rule</u>.

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & -2 \\ 1 & -4 & 0 \end{vmatrix} = -3\begin{vmatrix} 2 & 3 \\ -4 & 0 \end{vmatrix} + 0 - (-2)\begin{vmatrix} 1 & 2 \\ 1 & -4 \end{vmatrix} = -36 - 12 = -48$$

$$\det(A(1)) = \begin{vmatrix} 13 & 2 & 3 \\ 2 & 0 & -2 \\ 16 & -4 & 0 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 \\ -4 & 0 \end{vmatrix} + 0 - (-2)\begin{vmatrix} 13 & 2 \\ 16 & -4 \end{vmatrix} = -24 - 168 = -192$$

$$\det(A(2)) = \begin{vmatrix} 1 & 13 & 3 \\ 3 & 2 & -2 \\ 1 & 16 & 0 \end{vmatrix} = \begin{vmatrix} 13 & 3 \\ 2 & -2 \end{vmatrix} - 16\begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} + 0 = -32 + 176 = 144$$

$$\det(A(3)) = \begin{vmatrix} 1 & 2 & 13 \\ 3 & 0 & 2 \\ 1 & -4 & 16 \end{vmatrix} = -3\begin{vmatrix} 2 & 13 \\ -4 & 16 \end{vmatrix} + 0 - 2\begin{vmatrix} 1 & 2 \\ 1 & -4 \end{vmatrix} = -252 + 12 = -240$$

$$x = \frac{\det(A(1))}{\det(A)} = \frac{-192}{-48} = 4 \qquad y = \frac{\det(A(2))}{\det(A)} = \frac{144}{-48} = -3$$

$$z = \frac{\det(A(3))}{\det(A)} = \frac{-240}{-48} = 5$$

Hence the solutions is $(4, -3, 5)$

b) Find the inverse of the matrix of coefficients <u>using the adjoint</u>, and use it to solve the system of linear equations.

$$\operatorname{cof}(A) = \begin{bmatrix} -8 & -2 & -12 \\ -12 & -3 & 6 \\ -4 & 11 & -6 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} -8 & -12 & -4 \\ -2 & -3 & 11 \\ -12 & 6 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \begin{bmatrix} \frac{1}{6} & \frac{1}{4} & \frac{1}{12} \\ \frac{1}{24} & \frac{1}{16} & \frac{-11}{48} \\ \frac{1}{4} & \frac{-1}{8} & \frac{1}{8} \end{bmatrix}$$

$$X = A^{-1}b = \begin{bmatrix} \frac{1}{6} & \frac{1}{4} & \frac{1}{12} \\ \frac{1}{24} & \frac{1}{16} & \frac{-11}{48} \\ \frac{1}{4} & \frac{-1}{8} & \frac{1}{8} \end{bmatrix}\begin{bmatrix} 13 \\ 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$$

Hence the solutions is $(4, -3, 5)$

Question 3 (3 points)

Prove that if A and B are matrices such that $ABA = I$ then B is an invertible matrix.

$$ABA = I$$

$$\det(ABA) = \det(I)$$

$$\det(A)\det(B)\det(A) = 1$$

$$\left[\det(A)\right]^2 \det(B) = 1$$

If $\det(B) = 0$, then $\left[\det(A)\right]^2 \det(B) = 0 \neq 1$, thus $\det(B) \neq 0$, hence B must be invertible.

Question 4 (10 points)

Let $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 4 & 7 \\ -5 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & 4 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix}$.

Using Maple, evaluate the following.

a) $(A+B)^2$

b) $A^2 + 2AB + B^2$

c) $\left(CB^{-1}A\right)^T$

d) $\det\left(CC^T\right)$

e) $\operatorname{tr}\left(\operatorname{Adj}(A)\right)$

Question 5 (4 points)

Solve the following systems of linear equations using Maple.

a)
$$\begin{aligned} x + 2y + 3z &= 4 \\ 5x + 6y + 7z &= 8 \\ 3x + 5y &= -1 \end{aligned}$$

b)
$$\begin{aligned} v + 2w + 3y - z &= 2 \\ 3v + 6w + 5x + y &= 1 \\ -3v - 6w - 10z + 7y - 3z &= 4 \end{aligned}$$

Question 6 (8 points)

Consider a simple economy consisting of three sectors: food, clothing, and shelter. The production of 1 unit of food requires the consumption of 0.4 units of food, 0.2 units of clothing, and 0.2 units of shelter. The production of 1 unit of clothing requires the consumption of 0.1 units of food, 0.3 units of clothing, and 0.3 units of shelter. The production of 1 unit of shelter requires the consumption of 0.3 units of food, 0.1 units of clothing, and 0.1 units of shelter. Find the level of production for each sector in order to satisfy the demand for \$140 million worth of food, \$28 million worth of clothing, and \$70 million worth of shelter.

$$C = \begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.1 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \quad D = \begin{bmatrix} 140 \\ 28 \\ 70 \end{bmatrix}$$

$$(I - C)X = D$$

$$\left[\begin{array}{ccc|c} 0.6 & -0.1 & -0.3 & 140 \\ -0.2 & 0.7 & -0.1 & 28 \\ -0.2 & -0.3 & 0.9 & 70 \end{array}\right] \xrightarrow[R_3 \to 3R_3 + R_1]{R_2 \to 3R_2 + R_1} \left[\begin{array}{ccc|c} 0.6 & -0.1 & -0.3 & 140 \\ 0 & 2 & -0.6 & 224 \\ 0 & -1 & 2.4 & 350 \end{array}\right]$$

$$\xrightarrow{R_3 \to 2R_3 + R_2} \left[\begin{array}{ccc|c} 0.6 & -0.1 & -0.3 & 140 \\ 0 & 2 & -0.6 & 224 \\ 0 & 0 & 4.2 & 924 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{4.2} R_3]{R_1 \to \frac{1}{0.6} R_1,\ R_2 \to \frac{1}{2} R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{6} & -\frac{1}{2} & \frac{700}{3} \\ 0 & 1 & -\frac{3}{10} & 112 \\ 0 & 0 & 1 & 220 \end{array}\right]$$

Hence $z = 220$,

$y = 112 + \frac{3}{10} 220 = 178$

$x = \frac{140}{3} + \frac{1}{2} 178 + \frac{1}{6} 220 = 373$.

Thus the level of production for food should be \$373 million, for clothing \$178 million and for shelter 220 million.

Question 7 (10 points)

A television poll was conducted among regular viewers of the national news in Quebec where the three national networks share the same time slot for the evening news. Results of the poll indicate that 50% of the viewers watch the CTV evening new, 40% watch the CBC evening news, and 10% watch the Global evening news. Furthermore, it was found that of those viewers who watched CTV evening news during 1 week, 85% would again watch the CTV evening news during the next week, 10% would watch the CBC news, and 5% would watch the Global news. Of those viewers who watched CBC evening news during 1 week, 85% would again watch the CBC evening news during the next week, 10% would watch the CTV news, and 5% would watch the Global news. Of those viewers who watched Global evening news during 1 week, 80% would again watch the Global evening news during the next week, 15% would watch the CTV news, and 5% would watch the CBC news.

a) What share of the audience consisting of regular viewers of the national news will each network command after 2 weeks?

$$X^{(1)} = PX^{(0)} = \begin{bmatrix} 0.85 & 0.10 & 0.15 \\ 0.10 & 0.85 & 0.05 \\ 0.05 & 0.05 & 0.80 \end{bmatrix} \begin{bmatrix} 0.50 \\ 0.40 \\ 0.10 \end{bmatrix} = \begin{bmatrix} 0.480 \\ 0.395 \\ 0.125 \end{bmatrix}$$

$$X^{(2)} = PX^{(1)} = \begin{bmatrix} 0.85 & 0.10 & 0.15 \\ 0.10 & 0.85 & 0.05 \\ 0.05 & 0.05 & 0.80 \end{bmatrix} \begin{bmatrix} 0.480 \\ 0.395 \\ 0.125 \end{bmatrix} = \begin{bmatrix} 0.46625 \\ 0.39000 \\ 0.14375 \end{bmatrix}$$

Hence the share of audience will be 46.7% for CTV, 39.0% for CBC and 14.3% for Global.

b) What share of the audience consisting of regular viewers of the national news will each network command after 4 weeks? One year? (Use Maple)

After 4 weeks: 45.1% for CTV, 38.1% for CBC and 16.8% for Global

After 1 year: 44.0% for CTV, 36.0% for CBC and 20.0% for Global

c) In the long run, what share of the audience will each network command?

$$(I-P)X = 0$$

$$\left[\begin{array}{ccc|c} 0.15 & -0.10 & -0.15 & 0 \\ -0.10 & 0.15 & -0.05 & 0 \\ -0.05 & -0.05 & 0.20 & 0 \end{array}\right] \xrightarrow[R_3 \to 3R_3 + R_1]{R_2 \to 3R_2 + 2R_1} \left[\begin{array}{ccc|c} 0.15 & -0.10 & -0.15 & 0 \\ 0 & 0.25 & -0.45 & 0 \\ 0 & -0.25 & 0.45 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 0.15 & -0.10 & -0.15 & 0 \\ 0 & 0.25 & -0.45 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{0.25}R_2]{R_1 \to \frac{1}{0.15}R_1} \left[\begin{array}{ccc|c} 1 & \frac{-2}{3} & -1 & 0 \\ 0 & 1 & \frac{-9}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = t \quad y = \frac{9}{5}t \quad x = \frac{11}{5}t$

Since $x + y + z = \frac{11}{5}t + \frac{9}{5}t + 1 = 5t = 1$, then $t = \frac{1}{5}$, so the solution is $X = \left(\frac{11}{25}, \frac{9}{25}, \frac{1}{5}\right)$.

Ergo, the share of audience is: 44% for CTV, 36% for CBC and 20% for Global