MATHEMATICS 201-105-RE
Linear Algebra
Martin Huard
Winter 2006

Assignment #1
SOLUTIONS

This assignment is due ***Wednesday February 1, 2006***.

Question 1 (10 points)

Simplify (if possible).

a) $3\begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} -6 & -21 & 27 \\ 87 & -9 & 36 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ -2 & 0 & 8 \end{bmatrix} = \begin{bmatrix} -8 & -25 & 21 \\ 89 & -9 & 28 \end{bmatrix}$

b) $tr\left(\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}^2\right) = tr\left(\begin{bmatrix} 4 & -3 & 11 \\ 0 & 1 & 16 \\ 0 & 0 & 9 \end{bmatrix}\right) = 4+1+9 = 14$

Question 2 (10 points)

Solve the following system of linear equations using either Gaussian elimination or the Gauss-Jordan method.

$$\begin{aligned} 2x \qquad\; + 18z + 7w &= 14 \\ x - 2y + 3z + w &= 4 \\ -3x + 8y - 3z + 2w &= -4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 18 & 7 & 14 \\ 1 & -2 & 3 & 1 & 4 \\ -3 & 8 & -3 & 2 & -4 \end{array}\right] \xrightarrow[R_3 \to 2R_3 + 3R_1]{R_2 \to 2R_2 - R_1} \left[\begin{array}{cccc|c} 2 & 0 & 18 & 7 & 14 \\ 0 & -4 & -12 & -5 & -6 \\ 0 & 16 & 48 & 25 & 34 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 4R_2} \left[\begin{array}{cccc|c} 2 & 0 & 18 & 7 & 14 \\ 0 & -4 & -12 & -5 & -6 \\ 0 & 0 & 0 & 5 & 10 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{5}R_3]{R_1 \to \frac{1}{2}R_1,\; R_2 \to \frac{-1}{4}R_2} \left[\begin{array}{cccc|c} 1 & 0 & 9 & \frac{7}{2} & 7 \\ 0 & 1 & 3 & \frac{5}{4} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$w = 2$

$z = t$

$y = \frac{3}{2} - \frac{5}{4}2 - 3t = -1 - 3t$

$x = 7 - \frac{7}{2}2 - 9t = -9t$

Hence the solution is $(-9t, -1-3t, t, 2)$

Question 3 (10 points)

Solve the following system equations.

$$\frac{1}{x} + \frac{2}{y} - \frac{4}{z} = 1$$

$$\frac{2}{x} + \frac{3}{y} + \frac{8}{z} = 0$$

$$-\frac{1}{x} + \frac{9}{y} + \frac{10}{z} = 5$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 0 \\ -1 & 9 & 10 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & -2 \\ 0 & 11 & 6 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 11R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & -2 \\ 0 & 0 & 182 & -16 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + \frac{1}{182} R_2]{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 1 & \frac{-8}{91} \end{array}\right]$$

Thus $\frac{1}{z} = \frac{-8}{91}$

$\frac{1}{y} = 2 + 16\left(\frac{-8}{91}\right) = \frac{54}{91}$

$\frac{1}{x} = 1 - 2\left(\frac{54}{91}\right) + 4\left(\frac{-8}{91}\right) = \frac{-7}{13}$

and the solution is $\left(\frac{-13}{7}, \frac{91}{54}, \frac{-91}{8}\right)$.

Question 4 (10 points)

Consider the following system of linear equations.

$$\begin{aligned} x + y + az &= a \\ x + y + a^2z &= 2a \\ ax + a^2y + z &= 3a \end{aligned}$$

Find all values of a such that

a) the system has no solutions
b) the system has a unique solution
c) the system has an infinite number of solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 1 & 1 & a^2 & 2a \\ a & a^2 & 1 & 3a \end{array}\right] \xrightarrow[R_3 \to R_3 - aR_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 0 & 0 & a^2-a & a \\ 0 & a^2-a & 1-a^2 & 3a-a^2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 0 & a^2-a & 1-a^2 & 3a-a^2 \\ 0 & 0 & a^2-a & a \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{a^2-a}R_3]{R_2 \to \frac{1}{a^2-a}R_2} \left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 0 & 1 & \frac{1-a^2}{a^2-a} & \frac{3a-a^2}{a^2-a} \\ 0 & 0 & 1 & \frac{a}{a^2-a} \end{array}\right]$$

Illegal if $a^2 - a \neq 0$
that is if $a \neq 0,1$

Hence, if $a \neq 1$ and $a \neq 0$ then the system has a unique solution.
If $a = 0$, then we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} z &= 1 \\ y &= t \\ x &= -t \end{aligned}$$

so our system has an infinite number of solutions
If $a = 1$, then we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

so our system has no solutions.

Question 5 (10 points)

Consider the matrix $A=\begin{bmatrix}2 & -1 & 3\\ -1 & 1 & -2\\ 4 & 1 & 2\end{bmatrix}$. Find the inverse of A using Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|ccc}2 & -1 & 3 & 1 & 0 & 0\\ -1 & 1 & -2 & 0 & 1 & 0\\ 4 & 1 & 2 & 0 & 0 & 1\end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to 2R_2 + R_1} \left[\begin{array}{ccc|ccc}2 & -1 & 3 & 1 & 0 & 0\\ 0 & 1 & -1 & 1 & 2 & 0\\ 0 & 3 & -4 & -2 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|ccc}2 & -1 & 3 & 1 & 0 & 0\\ 0 & 1 & -1 & 1 & 2 & 0\\ 0 & 0 & -1 & -5 & -6 & 1\end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + 3R_3]{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc}2 & -1 & 0 & -14 & -18 & 3\\ 0 & 1 & 0 & 6 & 8 & -1\\ 0 & 0 & -1 & -5 & -6 & 1\end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc}2 & 0 & 0 & -8 & -10 & 2\\ 0 & 1 & 0 & 6 & 8 & -1\\ 0 & 0 & -1 & -5 & -6 & 1\end{array}\right]$$

$$\xrightarrow[R_3 \to -R_3]{R_1 \to \frac{1}{2} R_1} \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -4 & -5 & 1\\ 0 & 1 & 0 & 6 & 8 & -1\\ 0 & 0 & 1 & 5 & 6 & -1\end{array}\right]$$

Hence $A^{-1}=\begin{bmatrix}-4 & -5 & 1\\ 6 & 8 & -1\\ 5 & 6 & -1\end{bmatrix}$

Question 6 (10 points)

A matrix A is ***idempotent*** if $A^2 = A$.

a) Show that $A = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$ is idempotent.

$$A^2 = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} = A$$

b) Prove that if A and is idempotent, then A^T is also idempotent.

To prove: $\left(A^T\right)^2 = A^T$

$$\begin{aligned} LS &= \left(A^T\right)^2 \\ &= A^T A^T \\ &= (AA)^T \\ &= A^T \quad \text{since } A \text{ is idempotent} \\ &= RS \end{aligned}$$

c) Prove that if A is idempotent, then $(I-2A)^{-1} = I - 2A$.

To prove: $(I-2A)(I-2A) = I$

$$\begin{aligned} LS &= (I-2A)(I-2A) \\ &= I - 2AI - 2IA + 4A^2 \\ &= I - 4A + 4A \quad \text{since } A \text{ is idempotent} \\ &= I \\ &= RS \end{aligned}$$