MATHEMATICS 201-103-RE
Differential Calculus
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IV - Continuity

SOLUTIONS

1. Examine the continuity of f at $x=a$. If f is discontinuous at $x=a$, state the kind of discontinuity.

a) $f(x)=\begin{cases} \dfrac{x^2-x-6}{x-3} & x\neq 3 \\ 5 & x=3 \end{cases}$ at $x=3$

1. $f(3)=5$

2. $$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{x^2-x-6}{x-3}$$
$$=\lim_{x\to 3}\frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}} \qquad x\neq 3$$
$$=\lim_{x\to 3}(x+2)$$
$$=5$$

3. $\lim_{x\to 3} f(x)=5=f(3)$

Thus f is continuous at $x=3$.

b) $f(x)=\begin{cases} \dfrac{x^2+x-6}{x-3} & x\neq 3 \\ 5 & x=3 \end{cases}$ at $x=3$

1. $f(3)=5$

2. $$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}\frac{x^2+x-6}{x-3} \qquad x<3$$
$$=-\infty \qquad \text{case } \tfrac{b}{0}$$
$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}\frac{x^2+x-6}{x-3} \qquad x>3$$
$$=\infty \qquad \text{case } \tfrac{b}{0}$$

Thus $\lim_{x\to 3} f(x) \ \nexists$

Hence f has an infinite discontinuity at $x=3$.

c) $f(x)=\begin{cases} \dfrac{x^2-x-6}{x-3} & x\neq 3 \\ 3 & x=3 \end{cases}$ at $x=3$

1. $f(3)=3$
2. $\lim\limits_{x\to 3} f(x)=\lim\limits_{x\to 3}\dfrac{x^2-x-6}{x-3}$
$$=\lim_{x\to 3}\frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}} \qquad x\neq 3$$
$$=\lim_{x\to 3}(x+2)$$
$$=5$$
3. $\lim\limits_{x\to 3} f(x)=5\neq f(3)=3$

   Thus *f* has a removable discontinuity at $x=3$.

d) $f(x)=\begin{cases} x^2+1 & x\leq 2 \\ 3x-4 & x>2 \end{cases}$ at $x=2$

1. $f(2)=2^2+1=5$
2. $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}(x^2+1) \quad x<2$
$$=5$$
$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}(3x-4) \quad x>2$
$$=11$$
$\therefore \lim\limits_{x\to 2} f(x) \not\exists$

Thus *f* has a jump discontinuity at $x=2$.

e) $f(x)=\begin{cases} 4+x & x<1 \\ x^2+4 & x\geq 1 \end{cases}$ at $x=1$

1. $f(1)=1^2+4=5$
2. $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(4+x) \quad x<1$
$$=5$$
$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x^2+4) \quad x>1$
$$=5$$
$\therefore \lim\limits_{x\to 1} f(x)=5$
3. $\lim\limits_{x\to 1} f(x)=5=f(1)$

Thus *f* is continuous at $x=1$.

f) $f(x)=\begin{cases}\dfrac{|x-3|}{x-3} & x<3\\ \dfrac{1}{x-4} & x\geq 3\end{cases}$ at $x=3$

1. $f(3)=\dfrac{1}{3-4}=-1$

2. $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}\dfrac{|x-3|}{x-3} \quad x<3$

$= \lim\limits_{x\to 3^-}\dfrac{-(x-3)}{x-3}$

$= \lim\limits_{x\to 3^-} -1$

$=-1$

$\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}\dfrac{1}{x-4} \quad x>3$

$=-1$

$\therefore \lim\limits_{x\to 3} f(x)=-1$

3. $\lim\limits_{x\to 3} f(x)=-1=f(3)$

Thus *f* is continuous at $x=3$.

g) $f(x)=\begin{cases}3+x^2 & x<2\\ 0 & x=2\\ 11-x^2 & x>2\end{cases}$ at $x=2$

1. $f(2)=0$

2. $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\left(3+x^2\right) \quad x<2$

$=7$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\left(11-x^2\right) \quad x>2$

$=7$

$\therefore \lim\limits_{x\to 2} f(x)=7$

3. $\lim\limits_{x\to 2} f(x)=7\neq f(2)=0$

Thus *f* has a removable discontinuity at $x=2$.

h) $f(x)=\begin{cases} [\![x]\!] & x\le 4 \\ \dfrac{x^2-5x+4}{x-4} & x>4 \end{cases}$ at $x=4$

1. $f(4)=[\![4]\!]=4$

2. $\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}[\![4]\!] \quad x<4$
$= 3$

$\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+}\dfrac{x^2-5x+4}{x-4} \quad x>4$
$=\lim_{x\to 4^+}\dfrac{(x-4)(x-1)}{x-4}$
$=\lim_{x\to 4^+}(x-1)$
$=3$

$\therefore \lim_{x\to 4} f(x)=3$

3. $\lim_{x\to 4} f(x)=3\ne f(4)=4$
Thus *f* has a removable discontinuity at $x=4$.

i) $f(x)=[\![1-x]\!]+[\![x-1]\!]$ at $x=1$

1. $f(1)=[\![1-1]\!]+[\![1-1]\!]=0+0=0$

2. $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left([\![1-x]\!]+[\![x-1]\!]\right) \quad x<1$
$=0+(-1)=-1$

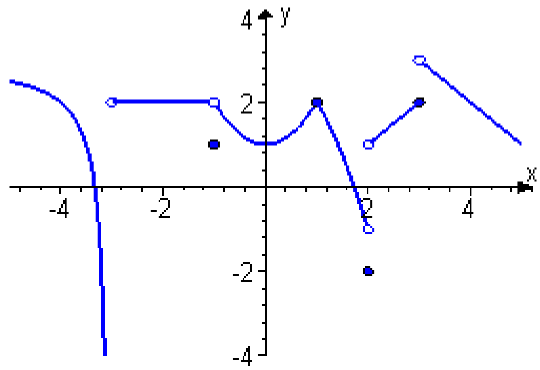
$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left([\![1-x]\!]+[\![x-1]\!]\right) \quad x>1$
$=-1+0=-1$

$\therefore \lim_{x\to 1} f(x)=-1$

3. $\lim_{x\to 1} f(x)=-1\ne f(1)=0$
Thus *f* has a removable discontinuity at $x=1$.

2. For each of the functions f whose graph is given below, find the points where f is discontinuous and state the type of discontinuity and find the intervals on which f is continuous.

a)



Since $\lim_{x\to -3^-} f(x) = -\infty$, then f has an infinite discontinuity at $x=-3$.

Since $\lim_{x\to -1} f(x) = 2 \neq f(-1) = 1$, then f has a removable discontinuity at $x=-1$

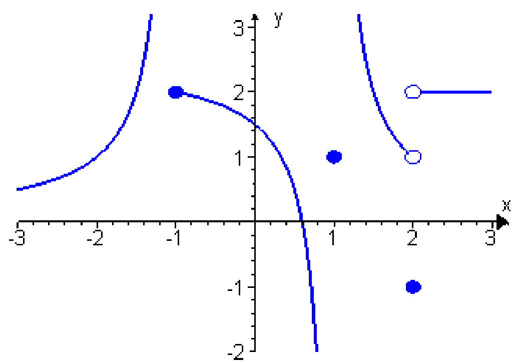
Since $\lim_{x\to 2^-} f(x) = -1 \neq \lim_{x\to 2^+} f(x) = 1$, then f has a jump discontinuity at $x=2$

Since $\lim_{x\to 3^-} f(x) = 2 \neq \lim_{x\to 3^+} f(x) = 3$, then f has a jump discontinuity at $x=3$

Since $\lim_{x\to 3^-} f(x) = 2 = f(3)$, then f is continuous from the left at $x=3$

Ergo, f is continuous on $(-\infty,-3),(-3,-1),(-1,2),(2,3],(3,\infty)$

b)



Since $\lim_{x\to -1^-} f(x) = -\infty$, then f has an infinite discontinuity at $x=-1$.

Since $\lim_{x\to 1^+} f(x) = \infty$, then f has an infinite discontinuity at $x=1$

Since $\lim_{x\to 2^-} f(x) = 1 \neq \lim_{x\to 2^+} f(x) = 2$, then f has a jump discontinuity at $x=2$

Since $\lim_{x\to -1^+} f(x) = 2 = f(-1)$, then f is continuous from the right at $x=-1$

Ergo, f is continuous on $(-\infty,-1),[-1,1),(1,2),(2,\infty)$

3. Discuss the continuity of the following functions. If the function is discontinuous at a point, state the kind of discontinuity.

a) $f(x)=\dfrac{x^2+x-6}{x^2-4}$

The function $f(x)$ is a rational function, thus is continuous on its domain $\mathbb{R}/\{-2,2\}$.

At $x=-2$,

$$\lim_{x\to-2^-}\frac{x^2+x-6}{x^2-4}=-\infty \quad \text{case } \tfrac{b}{0}$$

$$\lim_{x\to-2^+}\frac{x^2+x-6}{x^2-4}=\infty \quad \text{case } \tfrac{b}{0}$$

So f has an infinite discontinuity at $x=-2$.

At $x=2$,

$$\lim_{x\to2}\frac{x^2+x-6}{x^2-4}=\lim_{x\to2}\frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$$

$$=\lim_{x\to2}\frac{x+3}{x+2} \qquad x\neq2$$

$$=\tfrac{5}{4}$$

So f has a removable discontinuity $x=2$.

Hence f is continuous on $(-\infty,-2)$, $(-2,2)$ and $(2,\infty)$ with a removable discontinuity at $x=2$ and an infinity discontinuity at $x=-2$.

b) $f(x)=\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x^2-1}$

The function $f(x)$ is the sum of three rational functions, $\frac{1}{x}$ which is continuous on its domain $\mathbb{R}/\{0\}$, $\frac{1}{x-1}$ which is continuous on its domain $\mathbb{R}/\{1\}$ and $\frac{1}{x^2-1}$ which is continuous on its domain $\mathbb{R}/\{-1,1\}$. Thus $f(x)$ is continuous on $\mathbb{R}/\{-1,0,1\}$.

At $x=-1$,

$$\lim_{x\to-1^-}f(x)=\lim_{x\to-1^-}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x^2-1}\right)=-\tfrac{3}{2}+\lim_{x\to-1^-}\frac{1}{x^2-1}=\infty \quad \text{case } \tfrac{b}{0}$$

$$\lim_{x\to-1^+}f(x)=\lim_{x\to-1^+}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x^2-1}\right)=-\tfrac{3}{2}+\lim_{x\to-1^+}\frac{1}{x^2-1}=-\infty \quad \text{case } \tfrac{b}{0}$$

So f has an infinite discontinuity at $x=-1$.

At $x = 0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1}\right) = \lim_{x\to 0^-}\frac{1}{x} - 2 = -\infty \qquad \text{case } \tfrac{b}{0}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1}\right) = \lim_{x\to 0^+}\frac{1}{x} - 2 = \infty \qquad \text{case } \tfrac{b}{0}$$

So f has an infinite discontinuity at $x = 0$.

At $x = 1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1}\right) = 1 + \lim_{x\to 1^-}\frac{x+1+1}{x^2-1} = -\infty \quad \text{case } \tfrac{b}{0}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1}\right) = 1 + \lim_{x\to 1^+}\frac{x+1+1}{x^2-1} = \infty \qquad \text{case } \tfrac{b}{0}$$

So f has an infinite discontinuity at $x = 1$.

Ergo, f is continuous on $(-\infty,-1)$, $(-1,0)$, $(0,1)$ and $(1,\infty)$ with infinity discontinuities at $x = 0, \pm 1$

a) $f(x) = \sqrt{\dfrac{1}{x^2+1}}$

The function $f(x)$ is continuous on $\mathbb{R}$ since it is the composition of two continuous functions, $\dfrac{1}{x^2+1}$, a rational function continuous on its domain $\mathbb{R}$, and $\sqrt{x}$ the square root function which is also continuous since $\dfrac{1}{x^2+1} > 0$ for all values of x.

b) $f(x) = \left|x^2 + 2x + 1\right|$

The function $f(x)$ is continuous on $\mathbb{R}$ since it is the composition of two continuous function on $\mathbb{R}$, $x^2 + 2x + 1$, a polynomial continuous on its domain $\mathbb{R}$, and $|x|$ the absolute value function also continuous on its domain $\mathbb{R}$.

e) $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x-3} & x \neq 3 \\ 3 & x = 3 \end{cases}$

On $\mathbb{R}/\{3\}$, $f(x) = \dfrac{x^2 - x - 6}{x-3}$ which is a rational function, thus continuous on its domain $\mathbb{R}/\{3\}$.

At $x = 3$

1. $f(3) = 3$

2. $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \frac{x^2 - x - 6}{x-3}$

$= \lim\limits_{x\to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}} \qquad x \neq 3$

$= \lim\limits_{x\to 3} (x+2)$

$= 5$

3. $\lim\limits_{x\to 3} f(x) = 5 \neq f(3) = 3$

Thus *f* has a removable discontinuity at $x = 3$.

Ergo, *f* is continuous on $(-\infty, 3)$ and $(3, \infty)$ with a removable discontinuity at $x = 3$

f) $f(x) = \begin{cases} \frac{x}{x^2+1} & x \leq 1 \\ \frac{1}{x-5} & x > 1 \end{cases}$

On $(-\infty, 1)$, $f(x) = \frac{x}{x^2+1}$ which is a rational function, hence continuous on its domain $\mathbb{R}$, thus continuous on $(-\infty, 1)$.

On $(1, \infty)$, $f(x) = \frac{1}{x-5}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{5\}$, thus continuous on $(1, 5)$ and $(5, \infty)$.

At $x = 5$

$\lim\limits_{x\to 5^-} f(x) = \lim\limits_{x\to 5^-} \frac{1}{x-5} = -\infty \qquad \text{case } \frac{b}{0}$

$\lim\limits_{x\to 5^+} f(x) = \lim\limits_{x\to 5^+} \frac{1}{x-5} = \infty \qquad \text{case } \frac{b}{0}$

$\therefore \lim\limits_{x\to 5} f(x) \ \nexists$

Thus *f* has an infinite discontinuity at $x = 5$

At $x = 1$

1. $f(1) = \frac{1}{1^2+1} = \frac{1}{2}$

2. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \frac{x}{x^2+1} \qquad x < 1$

$= \frac{1}{2}$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \frac{1}{x-5} \qquad x > 1$

$= \frac{-1}{4}$

$\therefore \lim\limits_{x\to 1} f(x) \ \nexists$

Thus *f* has a jump discontinuity at $x = 1$, but since $\lim\limits_{x\to 1^-} f(x) = \frac{1}{2} = f(1)$, then *f* is continuous from the left at $x = 1$.

Hence *f* is continuous on $(-\infty,1]$, $(1,5)$ and $(5,\infty)$ with a jump discontinuity at $x=1$ and an infinite discontinuity at $x=5$.

g) $f(x)=\begin{cases} x^2-1 & x<3 \\ 2x+2 & x\geq 3 \end{cases}$

On $(-\infty,3)$, $f(x)=x^2-1$ which is a polynomial function, hence continuous on $\mathbb{R}$, thus continuous on $(-\infty,3)$.

On $(3,\infty)$, $f(x)=2x+2$ which is a polynomial function, hence continuous on $\mathbb{R}$, thus continuous on $(3,\infty)$.

At $x=3$

1. $f(3)=2\cdot 3+2=8$
2. $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(x^2-1) \quad x<3$ $\qquad$ $\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(2x+2) \quad x>3$
   $=8$ $\qquad$ $=8$
   $\therefore \lim_{x\to 3} f(x)=8$
3. $\therefore \lim_{x\to 3} f(x)=8=f(3)$

Thus *f* is continuous at at $x=3$.

Hence *f* is continuous on $\mathbb{R}$.

h) $f(x)=\begin{cases} 3x+1 & x<-2 \\ \dfrac{x^2-1}{x} & x\geq -2 \end{cases}$

On $(-\infty,-2)$, $f(x)=3x+1$ which is a polynomial function, hence continuous on $\mathbb{R}$, thus continuous on $(-\infty,-2)$.

On $(-2,\infty)$, $f(x)=\dfrac{x^2-1}{x}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{0\}$, thus continuous on $(-2,0),(0,\infty)$.

At $x=0$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{x^2-1}{x}=\infty \qquad \text{case } \tfrac{b}{0}$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x^2-1}{x}=-\infty \qquad \text{case } \tfrac{b}{0}$$

$$\therefore \lim_{x\to 0} f(x) \ \nexists$$

Thus *f* has an infinite discontinuity at $x=0$

At $x=-2$

1. $f(-2)=\frac{(-2)^2-1}{-2}=\frac{-3}{2}$

2. $\lim\limits_{x\to-2^-} f(x) = \lim\limits_{x\to-2^-}(3x+1) \quad x<-2$ $\qquad$ $\lim\limits_{x\to-2^+} f(x) = \lim\limits_{x\to-2^+}\frac{x^2-1}{x} \quad x>-2$

$= -5$ $\qquad$ $= \frac{-3}{2}$

$\therefore \lim\limits_{x\to3} f(x) \nexists$

Hence f has a jump discontinuity at $x=-2$

Since $\lim\limits_{x\to-2^+} f(x) = \frac{-3}{2} = f(-2)$, then f is continuous from the right at $x=-2$

Ergo, f is continuous on $(-\infty,-2),[-2,0),(0,\infty)$ with a jump discontinuity at $x=-2$ and an infinite discontinuity at $x=0$

i) $f(x) = \begin{cases} \dfrac{x^2+4}{x^2+2} & x\le 0 \\ \dfrac{x^2-4}{x^2-2x} & x>0 \end{cases}$

On $(-\infty,0)$, $f(x)=\frac{x^2+4}{x^2+2}$ which is a rational function, hence continuous on its domain $\mathbb{R}$, thus continuous on $(-\infty,0)$.

On $(0,\infty)$, $f(x)=\frac{x^2-4}{x^2-2x}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{0,2\}$, thus continuous on $(0,2),(2,\infty)$.

At $x=2$

$$\lim_{x\to2} f(x) = \lim_{x\to2}\frac{x^2-4}{x^2-2x}$$
$$= \lim_{x\to2}\frac{(x-2)(x+2)}{x(x-2)}$$
$$= \lim_{x\to2}\frac{x+2}{x} \qquad x\ne2$$
$$= 2$$

Thus f has a removable discontinuity at $x=2$

At $x=0$

1. $f(0)=\frac{0+4}{0+2}=2$

2. $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to-2^-}\frac{x^2+4}{x^2+2} \quad x<0$ $\qquad$ $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+}\frac{x^2-4}{x^2-2x} \quad x>0$

$=2$ $\qquad$ $=\infty$ $\qquad$ case $\frac{b}{0}$

$\therefore \lim\limits_{x\to3} f(x) \nexists$

Hence f has an infinite discontinuity at $x=0$

Since $\lim\limits_{x\to0^-} f(x) = 2 = f(0)$, then f is continuous from the left at $x=0$

Ergo, f is continuous on $(-\infty,0],(0,2),(2,\infty)$ with an infinite discontinuity at $x=0$ and a removable discontinuity at $x=2$.

j) $f(x)=\begin{cases}\dfrac{1}{x+2} & x\le 1\\ \dfrac{\sqrt{x}-2}{x-4} & x>1\end{cases}$

On $(-\infty,1)$, $f(x)=\frac{x^2+4}{x^2+2}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{-2\}$, thus continuous on $(-\infty,-2),(-2,1)$.

At $x=-2$

$$\lim_{x\to-2^-} f(x)=\lim_{x\to-2^-}\frac{1}{x+2}=-\infty \qquad \text{case } \tfrac{b}{0}$$

$$\lim_{x\to-2^+} f(x)=\lim_{x\to-2^+}\frac{1}{x+2}=\infty \qquad \text{case } \tfrac{b}{0}$$

$$\therefore \lim_{x\to-2} f(x) \;\nexists$$

Thus *f* has an infinite discontinuity at $x=-2$

On $(1,\infty)$, $f(x)=\frac{x^2-4}{x^2-2x}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{4\}$, thus continuous on $(1,4),(4,\infty)$.

At $x=4$

$$\lim_{x\to4} f(x)=\lim_{x\to2}\frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$=\lim_{x\to2}\frac{x-4}{(x-4)\left(\sqrt{x}+2\right)}$$

$$=\lim_{x\to2}\frac{1}{\sqrt{x}+2} \qquad x\ne 4$$

$$=\tfrac{1}{4}$$

Thus *f* has a removable discontinuity at $x=4$

At $x=1$

1. $f(1)=\frac{1}{1+2}=\frac{1}{3}$
2. $\lim_{x\to1^-} f(x)=\lim_{x\to1^-}\frac{1}{x+2} \quad x<0$ $\qquad$ $\lim_{x\to1^+} f(x)=\lim_{x\to1^+}\frac{\sqrt{x}-2}{x-4} \quad x>1$

   $=\frac{1}{3}$ $\qquad$ $=\frac{1}{3}$

   $\therefore \lim_{x\to3} f(x)=\frac{1}{3}$
3. $\lim_{x\to1} f(x)=\frac{1}{3}=f(1)$

Hence f is continuous at $x=1$

Ergo, *f* is continuous on $(-\infty,-2),(-2,4),(4,\infty)$ with an infinite discontinuity at $x=-2$ and a removable discontinuity at $x=4$.

k) $f(x)=\begin{cases}\dfrac{x}{x-1} & x\le -1\\ \sqrt{x+1} & -1<x<4\\ \dfrac{2x-10}{x^2-7x+10} & x\ge 4\end{cases}$

On $(-\infty,-1)$, $f(x)=\dfrac{x}{x-1}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{1\}$, thus continuous on $(-\infty,-1)$.

On $(-1,4)$, $f(x)=\sqrt{x+1}$ which is the composition of a root function and a polynomial, thus continuous on its domain $[-1,\infty)$, thus continuous on $(-1,4)$.

On $(4,\infty)$, $f(x)=\dfrac{2x-10}{x^2-7x+10}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{2,5\}$, thus continuous on $(4,5)$ and $(5,\infty)$.

At $x=5$

$$\begin{aligned}\lim_{x\to5} f(x) &= \lim_{x\to5}\frac{2x-10}{x^2-7x-10}\\ &= \lim_{x\to5}\frac{2\cancel{(x-5)}}{(x-2)\cancel{(x-5)}}\\ &= \lim_{x\to5}\frac{2}{x-2} \qquad x\neq5\\ &= \frac{2}{3}\end{aligned}$$

Thus f has a removable discontinuity at $x=5$

At $x=-1$

1. $f(-1)=\dfrac{-1}{-1-1}=\dfrac{1}{2}$

2. $\lim\limits_{x\to-1^-} f(x)=\lim\limits_{x\to-1^-}\dfrac{x}{x-1}=\tfrac{1}{2}$ $\quad x<-1$ $\qquad$ $\lim\limits_{x\to-1^+} f(x)=\lim\limits_{x\to-1^+}\sqrt{x+1}=0$ $\quad x>-1$

$\therefore \lim\limits_{x\to-1} f(x)\ \nexists$

Thus f has a jump discontinuity at $x=-1$, but since $\lim\limits_{x\to-1^-} f(x)=\tfrac{1}{2}=f(-1)$, then f is continuous from the left at $x=-1$.

At $x=4$

1. $f(4)=\dfrac{2\cdot4-10}{4^2-7\cdot4+10}=1$

2. $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \sqrt{x+1} \quad x<4$ $\qquad$ $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{2x-10}{x^2-7x+10} \quad x>4$

$= \sqrt{5}$ $\qquad$ $=1$

$\therefore \lim_{x\to 4} f(x) \nexists$

Thus f has a jump discontinuity at $x=4$, but since $\lim_{x\to 4^+} f(x) = 1 = f(4)$, then f is continuous from the right at $x=4$.

Hence f is continuous on $(-\infty,-1]$, $(-1,4)$, $[4,5)$ and $(5,\infty)$ with jump discontinuities at $x=-1,4$ and a removable discontinuity at $x=5$.

l) $$f(x) = \begin{cases} \frac{1}{x^2+2x} & x \le -1 \\ \frac{2}{x^2-x-2} & -1 < x \le 1 \\ \frac{x^2-3x+2}{x-1} & x > 1 \end{cases}$$

On $(-\infty,-1)$, $f(x) = \frac{1}{x^2+2x}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{-2,0\}$, thus continuous on $(-\infty,-2)$ and $(-2,-1)$.

At $x=-2$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x^2+2x} = \infty \qquad \text{case } \tfrac{b}{0}$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{1}{x^2+2x} = -\infty \qquad \text{case } \tfrac{b}{0}$$

$\therefore \lim_{x\to -2} f(x) \nexists$

Thus f has an infinite discontinuity at $x=-2$

On $(-1,1)$, $f(x) = \frac{2}{x^2-x-2}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{-1,2\}$, thus continuous on $(-1,1)$.

On $(1,\infty)$, $f(x) = \frac{x^2-3x+2}{x-1}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{1\}$, thus continuous on $(1,\infty)$.

At $x=-1$

1. $f(-1) = \frac{1}{(-1)^2+2(-1)} = -1$

2. $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{1}{x^2+2x} \quad x<-1$

$= -1$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{2}{x^2-x-2} \qquad x>-1$$
$$= \infty \qquad \text{case } \tfrac{b}{0}$$
$$\therefore \lim_{x\to -1} f(x) \ \nexists$$

Thus *f* has an infinite discontinuity at $x=-1$, but since $\lim_{x\to -1^-} f(x) = -1 = f(-1)$, then *f* is continuous from the left at $x=-1$.

At $x=1$

1. $f(1) = \frac{2}{1^2-1-2} = -1$
2. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{2}{x^2-x-2} = -1 \qquad x<1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-3x+2}{x-1} \qquad x>1$$
$$= \lim_{x\to 1^+} \frac{(x-2)\cancel{(x-1)}}{\cancel{x-1}}$$
$$= \lim_{x\to 1^+} (x-2)$$
$$= -1$$

$$\therefore \lim_{x\to 1} f(x) = -1$$

3. $\lim_{x\to 1} f(x) = -1 = f(1)$

Thus *f* is continuous at $x=1$.

Ergo, *f* is continuous on *f* is continuous on $(-\infty,-2)$, $(-2,-1]$ and $(-1,\infty)$ with infinite discontinuities at $x=-2,-1$.

m) $$f(x) = \begin{cases} \dfrac{x^2-16}{x^2+4x} & x<0 \\ \dfrac{\sqrt{x+1}-2}{x^2-9} & 0\le x\le 3 \\ \dfrac{1-\frac{3}{x}}{x^2+2x-15} & x>3 \end{cases}$$

On $(-\infty,0)$, $f(x) = \frac{x^2-16}{x^2+4x}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{-4,0\}$, thus continuous on $(-\infty,-4)$ and $(-4,0)$.

At $x=-4$

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} \frac{x^2-16}{x^2+4x}$$
$$= \lim_{x\to -4} \frac{(x-4)\cancel{(x+4)}}{x\cancel{(x+4)}}$$
$$= \lim_{x\to -4} \frac{x-4}{x}$$
$$= 2$$

Thus *f* has a removable discontinuity at $x=-4$

On $(0,3)$, $f(x)=\frac{\sqrt{x+1}-2}{x^2-9}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{-3,3\}$, thus continuous on $(0,3)$.

On $(3,\infty)$, $f(x)=\frac{1-\frac{3}{x}}{x^2+2x-15}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{-5,3\}$, thus continuous on $(3,\infty)$.

At $x=0$

1. $f(0)=\frac{\sqrt{0+1}-2}{0-9}=\frac{1}{9}$
2. $\lim_{x\to 0^-} f(x) = \lim_{x\to -1^-} \frac{x^2-16}{x^2+4x} \qquad x<0$
   $= \infty \qquad \text{case } \frac{b}{0}$
   $\lim_{x\to 0^+} f(x) = \lim_{x\to -1^+} \frac{\sqrt{x+1}-2}{x^2-9} \qquad x>0$
   $= \frac{1}{9}$
   $\therefore \lim_{x\to 0} f(x) \not\exists$

Thus *f* has an infinite discontinuity at $x=0$, but since $\lim_{x\to 0^+} f(x)=\frac{1}{9}=f(0)$, then *f* is continuous from the right at $x=0$.

At $x=3$

1. $f(3) \not\exists$
2. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{\sqrt{x+1}-2}{x^2-9}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \qquad x<3$
   $= \lim_{x\to 3^-} \frac{x-3}{(x-3)(x+3)\left(\sqrt{x+1}+2\right)}$
   $= \lim_{x\to 3^-} \frac{1}{(x+3)\left(\sqrt{x+1}+2\right)}$
   $= \frac{1}{24}$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{1-\frac{3}{x}}{x^2+2x-15} \qquad x>3$$

$$= \lim_{x\to 3^+} \frac{x-3}{x(x+5)(x-3)}$$

$$= \lim_{x\to 3^+} \frac{1}{x(x+5)}$$

$$= \tfrac{1}{24}$$

$$\therefore \lim_{x\to 1} f(x) = \tfrac{1}{24}$$

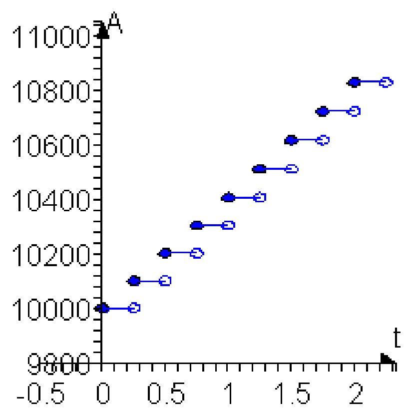
Thus f has a removable discontinuity at $x=3$.

Ergo, f is continuous on $(-\infty,-4)$, $(-4,0)$, $[0,3)$ and $(3,\infty)$ with a removable discontinuities at $x=-4$ and $x=3$, and an infinite discontinuity at $x=0$.

4. A deposit of \$10 000 is made in an account that pays 4% compounded quarterly. The amount A in the account after t years is

$$A = 10000(1.01)^{[\![4t]\!]}, \quad t \geq 0$$

a) Sketch the graph of A.



b) From the graph of A, find the intervals of continuity.

$$\left[0,\tfrac{1}{4}\right),\left[\tfrac{1}{4},\tfrac{1}{2}\right),\left[\tfrac{1}{2},\tfrac{3}{4}\right),\left[\tfrac{3}{4},1\right),\left[1,\tfrac{5}{4}\right),\ldots$$

c) What is the balance after 2 years?

$$A(2) = 10000(1.01)^{[\![8]\!]} = \$10828.57$$