MATHEMATICS 201-009-50
Precalculus
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XXI –Trigonometric Formulas

1. Find the exact value. (Do not use a calculator!)
   a) $\sin\left(\frac{5\pi}{12}\right)$ b) $\cos\frac{\pi}{12}$ c) $\sin\frac{17\pi}{12}$ d) $\tan\left(\frac{7\pi}{12}\right)$
   e) $\sec\left(\frac{-5\pi}{12}\right)$ f) $\sin 105°$ g) $\tan 15°$ h) $\csc 195°$
   i) $\sin\frac{\pi}{12}\cos\frac{5\pi}{12}-\sin\frac{5\pi}{12}\cos\frac{\pi}{12}$ j) $\cos\frac{7\pi}{9}\cos\frac{4\pi}{9}+\sin\frac{7\pi}{9}\sin\frac{4\pi}{9}$
   k) $\sin\frac{3\pi}{8}\cos\frac{7\pi}{8}+\sin\frac{7\pi}{8}\cos\frac{3\pi}{8}$ l) $\cos\frac{6\pi}{7}\cos\frac{\pi}{7}-\sin\frac{6\pi}{7}\cos\frac{\pi}{7}$
   m) $\sin\frac{\pi}{8}$ n) $\cos\frac{5\pi}{8}$ o) $\tan\frac{-\pi}{8}$ p) $\csc 22.5°$
   q) $\sin\frac{\pi}{12}-\sin\frac{5\pi}{12}$ r) $\cos\frac{7\pi}{12}+\cos\frac{11\pi}{12}$ s) $\sin\frac{17\pi}{12}+\sin\frac{13\pi}{12}$ t) $\cos\frac{13\pi}{12}-\cos\frac{7\pi}{12}$
   u) $\cos\frac{\pi}{24}\sin\frac{7\pi}{24}$ v) $\sin\frac{\pi}{4}\sin\frac{\pi}{12}$ w) $\sin 255°\cos 195°$ x) $\cos\frac{13\pi}{24}\sin\frac{7\pi}{24}$

2. If $\sin\alpha=\frac{3}{5}$ and $\cos\beta=\frac{-12}{13}$, where α and β are both in quadrant II, find the exact value of the trigonometric function.
   a) $\sin(\alpha+\beta)$ b) $\sin(\alpha-\beta)$ c) $\cos(\alpha+\beta)$ d) $\cos(\alpha-\beta)$
   e) $\tan(\alpha+\beta)$ f) $\csc(\alpha+\beta)$ g) $\sec(\alpha-\beta)$ h) $\cot(\alpha-\beta)$

3. If $\sin\alpha=\frac{1}{2}$ and $\cos\beta=\frac{1}{3}$, where α is in quadrant II and β is in quadrant IV, find the exact value of the trigonometric function.
   a) $\sin(\alpha+\beta)$ b) $\sin(\alpha-\beta)$ c) $\cos(\alpha+\beta)$ d) $\cos(\alpha-\beta)$
   e) $\tan(\alpha+\beta)$ f) $\csc(\alpha+\beta)$ g) $\sec(\alpha-\beta)$ h) $\cot(\alpha-\beta)$

4. If $\sin\theta=\frac{3}{5}$ where θ is in quadrant II, then find the exact value of the trigonometric function.
   a) $\sin 2\theta$ b) $\cos 2\theta$ c) $\sin\frac{\theta}{2}$ d) $\cos\frac{\theta}{2}$
   e) $\tan 2\theta$ f) $\csc\frac{\theta}{2}$ g) $\sec 2\theta$ h) $\cot\frac{\theta}{2}$

5. If $\tan\theta=-\sqrt{3}$ where θ is in quadrant IV, then find the exact value of the trigonometric function.
   a) $\sin 2\theta$ b) $\cos 2\theta$ c) $\sin\frac{\theta}{2}$ d) $\cos\frac{\theta}{2}$
   e) $\tan\frac{\theta}{2}$ f) $\sec\frac{\theta}{2}$ g) $\csc 2\theta$ h) $\cot 2\theta$

6. If $\sec\theta=2$ where θ is in quadrant IV, then find the exact value of the trigonometric function.
   a) $\sin 2\theta$ b) $\cos 2\theta$ c) $\sin\frac{\theta}{2}$ d) $\cos\frac{\theta}{2}$
   e) $\tan 2\theta$ f) $\csc\frac{\theta}{2}$ g) $\sec 2\theta$ h) $\cot\frac{\theta}{2}$

7. Expand $\cos 4x$ in terms of $\sin x$.

8. Expand $\sin 6x$ in terms of $\sin x$ and $\cos x$.

9. Express $\sin^4 x\cos^2 x$ in terms of the first power of cosine.

10. Express $\cos^6 x$ in terms of the first power of cosine.

11. Write the product as a sum.
    a) $\sin 4x\sin 5x$ b) $3\sin 2x\cos 3x$ c) $6\cos 3\theta\sin\theta$ d) $5\cos(\theta+\pi)\cos(\theta-\pi)$

12. Write the sum as a product.
    a) $\sin 4x+\sin 7x$ b) $\cos 3x+\cos 2x$ c) $\sin 5x-\sin 8x$ d) $\cos\frac{\theta}{2}-\cos\frac{\theta}{3}$

13. Verify the identity.
    a) $\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$ b) $\sin(5\pi-\theta)=\sin\theta$
    c) $\sin(\pi-\theta)+\cos\left(\frac{\pi}{2}+\theta\right)=0$ d) $\cos 2\theta=1-2\sin^2\theta$
    e) $\tan(2\pi-\theta)=-\tan\theta$ f) $\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$
    g) $\cot\left(\frac{3\pi}{2}+\theta\right)=-\tan\theta$ h) $\dfrac{\cos(x+y)}{\cos x\cos y}=1-\tan x\tan y$
    i) $\sin(x+y)+\sin(x-y)=2\sin x\cos y$ j) $\sec(A+B)=\dfrac{\csc A\csc B}{\cot A\cot B-1}$
    k) $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$ l) $\csc(A-B)=\dfrac{\csc A\csc B}{\cot B-\cot A}$
    m) $\cos^4\theta-\sin^4\theta=\cos 2\theta$ n) $\cos^2 2\theta-\sin^2 2\theta=\cos 4\theta$
    o) $\cot 2\alpha=\dfrac{1-\tan^2\alpha}{2\tan\alpha}$ p) $\dfrac{\sin 4x}{\sin x}=4\cos x\cos 2x$
    q) $\csc 2\theta=\frac{1}{2}\sec\theta\csc\theta$ r) $\cos\theta=\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
    s) $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$ t) $(\sin\theta+\cos\theta)^2=1+\sin 2\theta$
    u) $\tan\frac{x}{2}=\csc x-\cot x$ v) $\dfrac{\sin 3\theta+\sin 7\theta}{\cos 3\theta+\cos 7\theta}=\tan 5\theta$
    w) $\dfrac{\cos 4\theta-\cos 8\theta}{\cos 4\theta+\cos 8\theta}=\tan 2\theta\tan 6\theta$ x) $\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\tan\left(\dfrac{\alpha+\beta}{2}\right)$
    y) $1+\cos 2\theta+\cos 4\theta+\cos 6\theta=4\cos\theta\cos 2\theta\cos 3\theta$

ANSWERS

1. a) $\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$ b) $\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$ c) $-\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$ d) $-2-\sqrt{3}$
   e) $\sqrt{2}+\sqrt{6}$ f) $\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$ g) $2-\sqrt{3}$ h) $-\sqrt{2}-\sqrt{6}$
   i) $\frac{-\sqrt{3}}{2}$ j) $\frac{1}{2}$ k) $\frac{-\sqrt{2}}{2}$ l) -1
   m) $\frac{1}{2}\sqrt{2-\sqrt{2}}$ n) $\frac{-1}{2}\sqrt{2-\sqrt{2}}$ o) $1-\sqrt{2}$ p) $\sqrt{2\sqrt{2}+2}$
   q) $\frac{-\sqrt{2}}{2}$ r) $\frac{-\sqrt{6}}{2}$ s) $\frac{-\sqrt{6}}{2}$ t) $\frac{-\sqrt{2}}{2}$
   u) $\frac{\sqrt{2}}{4}+\frac{\sqrt{3}}{4}$ v) $\frac{\sqrt{3}}{4}-\frac{1}{4}$ w) $\frac{1}{2}+\frac{\sqrt{3}}{4}$ x) $\frac{\sqrt{2}}{4}-\frac{\sqrt{3}}{4}$
2. a) $\frac{-56}{65}$ b) $\frac{-16}{65}$ c) $\frac{33}{65}$ d) $\frac{63}{65}$
   e) $\frac{-56}{33}$ f) $\frac{-65}{56}$ g) $\frac{65}{63}$ h) $\frac{-63}{16}$
3. a) $\frac{1}{6}+\frac{\sqrt{6}}{3}$ b) $\frac{1}{6}-\frac{\sqrt{6}}{3}$ c) $\frac{\sqrt{2}}{3}-\frac{\sqrt{3}}{6}$ d) $-\frac{\sqrt{2}}{3}-\frac{\sqrt{3}}{6}$
   e) $\frac{8\sqrt{2}}{5}+\frac{9\sqrt{3}}{5}$ f) $\frac{-6}{23}+\frac{12\sqrt{6}}{23}$ g) $\frac{6\sqrt{3}}{5}-\frac{12\sqrt{2}}{5}$ h) $\frac{9\sqrt{3}}{23}+\frac{8\sqrt{2}}{23}$
4. a) $\frac{-24}{25}$ b) $\frac{7}{25}$ c) $\frac{3\sqrt{10}}{10}$ d) $\frac{\sqrt{10}}{10}$
   e) $\frac{-24}{7}$ f) $\frac{\sqrt{10}}{3}$ g) $\frac{25}{7}$ h) $\frac{1}{3}$
5. a) $\frac{-\sqrt{3}}{2}$ b) $\frac{-1}{2}$ c) $\frac{-1}{2}$ d) $\frac{\sqrt{3}}{2}$
   e) $\frac{-\sqrt{3}}{3}$ f) $\frac{2\sqrt{3}}{3}$ g) $\frac{-2\sqrt{3}}{3}$ h) $\frac{\sqrt{3}}{3}$
6. a) $\frac{-\sqrt{3}}{2}$ b) $\frac{-1}{2}$ c) $\frac{-1}{2}$ d) $\frac{\sqrt{3}}{2}$
   e) $\sqrt{3}$ f) -3 g) -2 h) $-\sqrt{3}$
7. $\cos 4x = 1-2\sin^2 2x = 1-2\left(2\sin x\cos x\right)^2 = 1-8\sin^2 x\cos^2 x = 1-8\sin^2 x\left(1-\sin^2 x\right)$
   $= 1-8\sin^2 x + 8\sin^4 x$
8. $\sin 6x = \sin\left(4x+2x\right) = \sin 4x\cos 2x + \sin 2x\cos 4x = 2\sin 2x\cos^2 2x + \sin 2x\left(1-2\sin^2 2x\right)$
   $= 4\sin x\cos x\left(\cos^2 x - \sin^2 x\right)^2 + 2\sin x\cos x - 2\left(2\sin x\cos x\right)^3$
   $= 4\sin x\cos^4 x - 24\sin^3 x\cos^3 x + 4\sin^5 x\cos x + 2\sin x\cos x$
9. $\sin^4 x\cos^2 x = \left(1-\cos^2 x\right)^2\cos^2 x = \cos^2 x - 2\cos^4 x + \cos^6 x$
10. $\cos^6 x = \left(1-\sin^2 x\right)^3 = 1-3\sin^2 x + 3\sin^4 x - \sin^6 x$
11. a) $\frac{1}{2}\cos x - \frac{1}{2}\cos 9x$ b) $\frac{3}{2}\sin 5x - \frac{3}{2}\sin x$ c) $3\sin 4\theta - 3\sin 2\theta$ d) $\frac{5}{2}\cos 2\theta + \frac{5}{2}$
12. a) $2\sin\frac{11x}{2}\cos\frac{3x}{2}$ b) $2\cos\frac{5x}{2}\cos\frac{x}{2}$ c) $-2\cos\frac{13x}{2}\sin\frac{3x}{2}$ d) $-2\sin\frac{5\theta}{12}\sin\frac{\theta}{12}$
13. a) $LS = \sin\left(\frac{\pi}{2}+\theta\right) = \sin\frac{\pi}{2}\cos\theta + \sin\theta\cos\frac{\pi}{2} = 1\cdot\cos\theta + \sin\theta\cdot 0 = \cos\theta = RS$
    b) $LS = \sin\left(5\pi-\theta\right) = \sin 5\pi\cos\theta - \sin\theta\cos 5\pi = 0\cdot\cos\theta - \sin\theta\cdot\left(-1\right) = \sin\theta = RS$
    c) $LS = \sin\left(\pi-\theta\right) + \cos\left(\frac{\pi}{2}+\theta\right) = \sin\pi\cos\theta - \sin\theta\cos\pi + \cos\frac{\pi}{2}\cos\theta - \sin\frac{\pi}{2}\sin\theta$
    $= \sin\theta - \sin\theta = 0 = RS$

d) $LS = \cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta = 1 - \sin^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta = RS$

e) $LS = \tan(2\pi - \theta) = \frac{\tan 2\pi - \tan\theta}{1 + \tan 2\pi \tan\theta} = \frac{0 - \tan\theta}{1 + 0 \cdot \tan\theta} = -\tan\theta = RS$

f) $LS = \csc\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\sin\frac{\pi}{2}\cos\theta - \sin\theta\cos\frac{\pi}{2}} = \frac{1}{\cos\theta} = \sec\theta = RS$

g) $LS = \cot\left(\frac{3\pi}{2} + \theta\right) = \frac{\cos\left(\frac{3\pi}{2} + \theta\right)}{\sin\left(\frac{3\pi}{2} + \theta\right)} = \frac{\cos\frac{3\pi}{2}\cos\theta - \sin\frac{3\pi}{2}\sin\theta}{\sin\frac{3\pi}{2}\cos\theta + \sin\theta\cos\frac{3\pi}{2}} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta = RS$

h) $LS = \frac{\cos(x+y)}{\cos x \cos y} = \frac{\cos x\cos y - \sin x \sin y}{\cos x \cos y} = \frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y} = 1 - \tan x \tan y = RS$

i) $LS = \sin(x+y) + \sin(x-y) = \sin x \cos y + \sin y \cos x + \sin x \cos y - \sin y \cos x$
$= 2\sin x \cos y = RS$

j) $LS = \sec(A+B) = \frac{1}{\cos(A+B)} = \frac{1}{\cos A \cos B - \sin A \sin B}$

$RS = \frac{\csc A \csc B}{\cot A \cot B - 1} = \frac{\frac{1}{\sin A \sin B}}{\frac{\cos A \cos B}{\sin A \sin B} - 1} = \frac{\frac{1}{\sin A \sin B}}{\frac{\cos A \cos B - \sin A \sin B}{\sin A \sin B}} = \frac{1}{\sin A \sin B}\frac{\sin A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{1}{\cos A \cos B - \sin A \sin B} = LS$

k) $LS = \cot(\alpha + \beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \sin\beta\cos\alpha}$

$RS = \frac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha} = \frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} - 1}{\frac{\cos\beta}{\sin\beta} + \frac{\cos\alpha}{\sin\alpha}} = \frac{\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\sin\alpha\sin\beta}} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\sin\beta}\frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta + \sin\beta\cos\alpha} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \sin\beta\cos\alpha} = LS$

l) $LS = \csc(A-B) = \frac{1}{\sin(A-B)} = \frac{1}{\sin A \cos B - \sin B \cos A}$

$RS = \frac{\csc A \csc B}{\cot B - \cot A} = \frac{\frac{1}{\sin A \sin B}}{\frac{\cos B}{\sin B} - \frac{\cos A}{\sin A}} = \frac{\frac{1}{\sin A \sin B}}{\frac{\sin A \cos B - \sin B \cos A}{\sin A \sin B}} = \frac{1}{\sin A \sin B}\frac{\sin A \sin B}{\sin A \cos B - \sin B \cos A} = \frac{1}{\sin A \cos B - \sin B \cos A} = LS$

m) $LS = \cos^4\theta - \sin^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 - \left(\frac{1-\cos 2\theta}{2}\right)^2 = \frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4} - \frac{1}{4} + \frac{\cos 2\theta}{2} - \frac{\cos^2 2\theta}{4} = \cos 2\theta = RS$

n) $LS = \cos^2 2\theta - \sin^2 2\theta = \frac{1+\cos 4\theta}{2} - \frac{1-\cos 4\theta}{2} = \cos 4\theta = RS$

o) $LS = \cot 2\alpha = \frac{1}{\tan 2\alpha} = \frac{1}{\frac{2\tan\alpha}{1-\tan^2\alpha}} = \frac{1-\tan^2\alpha}{2\tan\alpha} = RS$

p) $LS = \frac{\sin 4x}{\sin x} = \frac{2\sin 2x \cos 2x}{\sin x} = \frac{4\sin x \cos x \cos 2x}{\sin x} = 4\cos x \cos 2x = RS$

q) $LS = \csc 2\theta = \frac{1}{\sin 2\theta} = \frac{1}{2\sin\theta\cos\theta} = \frac{1}{2}\csc\theta\sec\theta = RS$

r) $RS = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\frac{1-\cos\theta}{1+\cos\theta}}{1+\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\frac{1+\cos\theta-1+\cos\theta}{1+\cos\theta}}{\frac{1+\cos\theta+1-\cos\theta}{1+\cos\theta}} = \frac{2\cos\theta}{1+\cos\theta}\frac{1+\cos\theta}{2} = \cos\theta = LS$

s) $LS = \tan 3\theta = \tan(\theta + 2\theta) = \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \frac{\tan\theta + \frac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta\frac{2\tan\theta}{1-\tan^2\theta}} = \frac{\frac{\tan\theta - \tan^3\theta + 2\tan\theta}{1-\tan^2\theta}}{\frac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}} = \frac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta}\frac{1-\tan^2\theta}{1-3\tan^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = RS$

t) $LS = (\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + \sin 2\theta = RS$

u) $LS = \tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \csc x - \cot x = RS$

v) $LS = \frac{\sin 3\theta + \sin 7\theta}{\cos 3\theta + \cos 7\theta} = \frac{2\sin 5\theta\cos(-4\theta)}{2\cos 5\theta\cos(-4\theta)} = \tan 5\theta = RS$

w) $LS = \frac{\cos 4\theta - \cos 8\theta}{\cos 4\theta + \cos 8\theta} = \frac{-2\sin 6\theta\sin(-2\theta)}{2\cos 6\theta\cos(-2\theta)} = \tan 6\theta\tan 2\theta = RS$

x) $LS = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} = \tan\frac{\alpha+\beta}{2} = RS$

y) $LS = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta = 1 + \cos 6\theta + \cos 2\theta + \cos 4\theta = 2\cos^2 3\theta + 2\cos 3\theta\cos(-\theta)$
$= 2\cos 3\theta(\cos 3\theta + \cos\theta) = 2\cos 3\theta(2\cos 2\theta\cos\theta) = 4\cos\theta\cos 2\theta\cos 3\theta = RS$