MATHEMATICS 201-009-50
Precalculus
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XX –Trigonometric Identities

1. Simplify the trigonometric expression.
   a) $\sin x \cot x$
   b) $\sec x \cot x \sin x$
   c) $\dfrac{\sec^2 x - 1}{\sec^2 x}$
   d) $\cot^2 x - \csc^2 x$
   e) $\dfrac{\sec\theta - 1}{1 - \cos\theta}$
   f) $\dfrac{\cos\theta}{\sec\theta} + \dfrac{\sin\theta}{\csc\theta}$
   g) $\tan^2\theta\cos^2\theta + \cot^2\theta\sin^2\theta$
   h) $\dfrac{\sin\theta}{1-\tan\theta} + \dfrac{\cos\theta}{1-\cot\theta}$

2. Verify the identity.
   a) $(1+\sin\theta)(1-\sin\theta) = \cos^2\theta$
   b) $\sin\theta + \cos\theta\cot\theta = \csc\theta$
   c) $\dfrac{1}{\cos x} - \dfrac{1}{\sec x} = \sec x - \cos x$
   d) $\dfrac{\tan^2\theta}{\sec\theta} = \sec\theta - \cos\theta$
   e) $\dfrac{1}{\tan x} + \tan x = \dfrac{\sec^2 x}{\tan x}$
   f) $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$
   g) $\sin^2\phi - \sin^4\phi = \cos^2\phi - \cos^4\phi$
   h) $\dfrac{\sin\theta\tan\theta}{1-\cos\theta} - 1 = \sec\theta$
   i) $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 2\sec\theta$
   j) $\dfrac{\sec\theta - 1}{\sec\theta + 1} = \dfrac{1-\cos\theta}{1+\cos\theta}$
   k) $1 - \dfrac{\sin^2\theta}{1+\cos\theta} = \cos\theta$
   l) $\dfrac{\csc\theta - \sec\theta}{\csc\theta\sec\theta} = \cos\theta - \sin\theta$
   m) $\dfrac{1+\cos\theta}{1-\cos\theta} - \dfrac{1-\cos\theta}{1+\cos\theta} = 4\csc\theta\cot\theta$
   n) $\sec\theta\csc\theta(\tan\theta + \cot\theta) = \sec^2\theta + \csc^2\theta$
   o) $\dfrac{\tan\beta - \cot\beta}{\tan\beta + \cot\beta} = \sin^2\beta - \cos^2\beta$
   p) $\dfrac{1}{\cos\phi + 1} + \dfrac{1}{\sec\phi + 1} = 1$
   q) $\dfrac{\cos\alpha - \cos\beta}{\sin\alpha + \sin\beta} + \dfrac{\sin\alpha - \sin\beta}{\cos\alpha + \cos\beta} = 0$
   r) $\dfrac{1-\cos\theta}{1+\cos\theta} = (\csc\theta - \cot\theta)^2$
   s) $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = 1 - \sin\theta\cos\theta$
   t) $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$
   u) $\dfrac{\sin^2\psi - \cos^2\psi}{1 - \cot^2\psi} = \sin^2\psi$
   v) $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta$
   w) $\tan x + \dfrac{\cos x}{1+\sin x} = \sec x$
   x) $\dfrac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

y) $\dfrac{\left(2\sin^2\theta-1\right)^2}{\sin^4\theta-\cos^4\theta}=1-2\cos^2\theta$

z) $\sec^4\theta-\tan^4\theta=1+2\tan^2\theta$

aa) $\dfrac{\cot\theta}{\csc\theta-1}=\dfrac{\csc\theta+1}{\cot\theta}$

bb) $\left(\tan\alpha+\tan\beta\right)\left(1-\cot\alpha\cot\beta\right)+\left(\cot\alpha+\cot\beta\right)\left(1-\tan\alpha\tan\beta\right)=0$

cc) $\left(\sin x-\tan x\right)\left(\cos x-\cot x\right)=\left(\cos x-1\right)\left(\sin x-1\right)$

dd) $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}=\sec\theta+\tan\theta$

ANSWERS

1. a) $\sin x \cot x = \sin x \frac{\cos x}{\sin x} = \cos x$

   b) $\sec x \cot x \sin x = \frac{1}{\cos x}\frac{\cos x}{\sin x}\sin x = 1$

   c) $\frac{\sec^2 x - 1}{\sec^2 x} = \frac{\frac{1}{\cos^2 x}-1}{\frac{1}{\cos^2 x}} = \frac{\frac{1-\cos^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}} = \frac{1-\cos^2 x}{\cos^2 x}\frac{\cos^2 x}{1} = 1-\cos^2 x = \sin^2 x$

   d) $\cot^2 x - \csc^2 x = \frac{\cos^2 x}{\sin^2 x} - \frac{1}{\sin^2 x} = \frac{\cos^2 x - 1}{\sin^2 x} = \frac{-\sin^2 x}{\sin^2 x} = -1$

   e) $\frac{\sec\theta - 1}{1-\cos\theta} = \frac{\frac{1}{\cos\theta}-1}{1-\cos\theta} = \frac{\frac{1-\cos\theta}{\cos\theta}}{1-\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$

   f) $\frac{\cos\theta}{\sec\theta} + \frac{\sin\theta}{\csc\theta} = \cos^2\theta + \sin^2\theta = 1$

   g) $\tan^2\theta\cos^2\theta + \cot^2\theta\csc^2\theta = \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta + \frac{\cos^2\theta}{\sin^2\theta}\sin^2\theta = \sin^2\theta + \cos^2\theta = 1$

   h) $\frac{\sin\theta}{1-\tan\theta} + \frac{\cos\theta}{1-\cot\theta} = \frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}} + \frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}} = \frac{\sin\theta}{\frac{\cos\theta-\sin\theta}{\cos\theta}} + \frac{\cos\theta}{\frac{\sin\theta-\cos\theta}{\sin\theta}} = \frac{\sin\theta\cos\theta}{\cos\theta-\sin\theta} - \frac{\sin\theta\cos\theta}{\cos\theta-\sin\theta} = 0$

2. a) $LS = (1+\sin\theta)(1-\sin\theta) = 1-\sin^2\theta = \cos^2\theta = RS$

   b) $LS = \sin\theta + \cos\theta\cot\theta = \sin\theta + \cos\theta\frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta+\cos^2\theta}{\sin\theta} = \frac{1}{\sin\theta} = \csc\theta = RS$

   c) $LS = \frac{1}{\cos x} - \frac{1}{\sec x} = \sec x - \frac{1}{\frac{1}{\cos x}} = \sec x - \cos x = RS$

   d) $LS = \frac{\tan^2\theta}{\sec\theta} = \frac{\sec^2\theta - 1}{\sec\theta} = \frac{\sec^2\theta}{\sec\theta} - \frac{1}{\sec\theta} = \sec\theta - \cos\theta = RS$

   e) $LS = \frac{1}{\tan x} + \tan x = \frac{1+\tan^2 x}{\tan x} = \frac{\sec^2 x}{\tan x} = RS$

   f) $LS = (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1 + \tan^2\theta - \tan^2\theta = 1 = RS$

   g) $LS = \sin^2\phi - \sin^4\phi = \sin^2\phi(1-\sin^2\phi) = (1-\cos^2\phi)\cos^2\phi = \cos^2\phi - \cos^4\phi = RS$

   h) $LS = \frac{\sin\theta\tan\theta}{1-\cos\theta} - 1 = \frac{\sin\theta\frac{\sin\theta}{\cos\theta}}{1-\cos\theta} - 1 = \frac{\sin^2\theta}{\cos\theta - \cos^2\theta} - 1 = \frac{\sin^2\theta - \cos\theta + \cos^2\theta}{\cos\theta - \cos^2\theta} = \frac{1-\cos\theta}{\cos\theta(1-\cos\theta)} = \frac{1}{\cos\theta} = \sec\theta = RS$

   i) $LS = \frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} = \frac{(1+\sin\theta)^2 + \cos^2\theta}{\cos\theta(1+\sin\theta)} = \frac{1+2\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1+\sin\theta)} = \frac{2+2\sin\theta}{\cos\theta(1+\sin\theta)} = \frac{2}{\cos\theta} = 2\sec\theta = RS$

   j) $LS = \frac{\sec\theta - 1}{\sec\theta + 1} = \frac{\frac{1}{\cos\theta}-1}{\frac{1}{\cos\theta}+1} = \frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{1+\cos\theta}{\cos\theta}} = \frac{1-\cos\theta}{\cos\theta}\cdot\frac{\cos\theta}{1+\cos\theta} = \frac{1-\cos\theta}{1+\cos\theta} = RS$

   k) $LS = 1 - \frac{\sin^2\theta}{1+\cos\theta} = \frac{1+\cos\theta - \sin^2\theta}{1+\cos\theta} = \frac{\cos^2\theta + \cos\theta}{1+\cos\theta} = \frac{\cos\theta(1+\cos\theta)}{1+\cos\theta} = \cos\theta = RS$

   l) $LS = \frac{\csc\theta - \sec\theta}{\csc\theta\sec\theta} = \frac{\frac{1}{\sin\theta} - \frac{1}{\cos\theta}}{\frac{1}{\sin\theta}\frac{1}{\cos\theta}} = \frac{\frac{\cos\theta - \sin\theta}{\sin\theta\cos\theta}}{\frac{1}{\sin\theta\cos\theta}} = \frac{\cos\theta-\sin\theta}{\sin\theta\cos\theta}\cdot\frac{\sin\theta\cos\theta}{1} = \cos\theta - \sin\theta = RS$

   m) $LS = \frac{1+\cos\theta}{1-\cos\theta} - \frac{1-\cos\theta}{1+\cos\theta} = \frac{(1+\cos\theta)^2 - (1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1+2\cos\theta+\cos^2\theta - 1 + 2\cos\theta - \cos^2\theta}{1-\cos^2\theta} = \frac{4\cos\theta}{\sin^2\theta} = 4\frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta}$

   $= 4\csc\theta\cot\theta = RS$

   n) $LS = \sec\theta\csc\theta(\tan\theta + \cot\theta) = \frac{1}{\cos\theta}\frac{1}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \sec^2\theta + \csc^2\theta = RS$

   o) $\frac{\tan\beta - \cot\beta}{\tan\beta + \cot\beta} = \frac{\frac{\sin\beta}{\cos\beta} - \frac{\cos\beta}{\sin\beta}}{\frac{\sin\beta}{\cos\beta} + \frac{\cos\beta}{\sin\beta}} = \frac{\frac{\sin^2\beta - \cos^2\beta}{\cos\beta\sin\beta}}{\frac{\sin^2\beta+\cos^2\beta}{\cos\beta\sin\beta}} = \frac{\sin^2\beta - \cos^2\beta}{\cos\beta\sin\beta}\frac{\cos\beta\sin\beta}{1} = \sin^2\beta - \cos^2\beta = RS$

   p) $LS = \frac{1}{\cos\phi+1} + \frac{1}{\sec\phi+1} = \frac{1}{\cos\phi+1} + \frac{1}{\frac{1}{\cos\phi}+1} = \frac{1}{\cos\phi+1} + \frac{1}{\frac{1+\cos\phi}{\cos\phi}} = \frac{1}{\cos\phi+1} + \frac{\cos\phi}{1+\cos\phi} = \frac{1+\cos\phi}{\cos\phi+1} = 1 = RS$

   q) $LS = \frac{\cos\alpha - \cos\beta}{\sin\alpha + \sin\beta} + \frac{\sin\alpha - \sin\beta}{\cos\alpha + \cos\beta} = \frac{\cos^2\alpha - \cos^2\beta + \sin^2\beta - \sin^2\beta}{(\sin\alpha+\sin\beta)(\cos\alpha+\cos\beta)} = \frac{1-1}{(\sin\alpha+\sin\beta)(\cos\alpha+\cos\beta)} = 0 = RS$

   r) $RS = (\csc\theta - \cot\theta)^2 = \csc^2\theta - 2\csc\theta\cot\theta + \cot^2\theta = \frac{1}{\sin^2\theta} - 2\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin^2\theta}$

   $= \frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta} = \frac{(1-\cos\theta)^2}{1-\cos^2\theta} = \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta} = LS$

s) $LS=\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=\frac{\sin\theta\left(1-\cos^2\theta\right)+\cos\theta\left(1-\sin^2\theta\right)}{\sin\theta+\cos\theta}=\frac{\sin\theta-\sin\theta\cos^2\theta+\cos\theta-\cos\theta\sin^2\theta}{\sin\theta+\cos\theta}=\frac{\sin\theta+\cos\theta-\sin\theta\cos\theta(\cos\theta+\sin\theta)}{\sin\theta+\cos\theta}$

$=\frac{(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)}{\sin\theta+\cos\theta}=1-\sin\theta\cos\theta=RS$

t) $LS=\frac{\tan x+\tan y}{\cot x+\cot y}=\frac{\tan x+\tan y}{\frac{1}{\tan x}+\frac{1}{\tan y}}=\frac{\tan x+\tan y}{\frac{\tan y+\tan x}{\tan x\tan y}}=\frac{\tan x+\tan y}{1}\cdot\frac{\tan x\tan y}{\tan y+\tan x}=\tan x\tan y=RS$

u) $LS=\frac{\sin^2\psi-\cos^2\psi}{1-\cot^2\psi}=\frac{\sin^2\psi-\cos^2\psi}{1-\frac{\cos^2\psi}{\sin^2\psi}}=\frac{\sin^2\psi-\cos^2\psi}{\frac{\sin^2\psi-\cos^2\psi}{\sin^2\psi}}=\frac{\sin^2\psi-\cos^2\psi}{1}\cdot\frac{\sin^2\psi}{\sin^2\psi-\cos^2\psi}=\sin^2\psi=RS$

v) $LS=\left(\tan\theta+\cot\theta\right)^2=\tan^2\theta+2\tan\theta\cot\theta+\cot^2\theta=\tan^2\theta+2\tan\theta\frac{1}{\tan\theta}+\cot^2\theta$

$=\tan^2\theta+2+\cot^2\theta=\tan^2\theta+1+1+\cot^2\theta=\sec^2\theta+\csc^2\theta=RS$

w) $LS=\tan x+\frac{\cos x}{1+\sin x}=\frac{\sin x}{\cos x}+\frac{\cos x}{1+\sin x}=\frac{\sin x(1+\sin x)+\cos^2x}{\cos x(1+\sin x)}=\frac{\sin x+\sin^2x+\cos^2x}{\cos x(1+\sin x)}=\frac{\sin x+1}{\cos x(1+\sin x)}=\frac{1}{\cos x}=\sec x=RS$

x) $LS=\frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}=\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}=\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}=\frac{1+\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\sec\theta+\tan\theta=RS$

y) $LS=\frac{\left(2\sin^2\theta-1\right)^2}{\sin^4\theta-\cos^4\theta}=\frac{\left(\sin^2\theta+\sin^2\theta-1\right)^2}{\sin^2\theta\left(1-\cos^2\theta\right)-\cos^2\theta\left(1-\sin^2\theta\right)}=\frac{\left(\sin^2\theta-\cos^2\theta\right)^2}{\sin^2\theta-\sin^2\theta\cos^2\theta-\cos^2\theta+\sin^2\theta\cos^2\theta}=\frac{\left(\sin^2\theta-\cos^2\theta\right)^2}{\sin^2\theta-\cos^2\theta}$

$=\sin^2\theta-\cos^2\theta=1-\cos^2\theta-\cos^2\theta=1-2\cos^2\theta=RS$

z) $LS=\sec^4\theta-\tan^4\theta=\sec^2\theta\left(1+\tan^2\theta\right)-\tan^2\theta\left(\sec^2-1\right)$

$=\sec^2\theta+\sec^2\theta\tan^2\theta-\tan^2\theta\sec^2\theta+\tan^2\theta=1+\tan^2\theta+\tan^2\theta=1+2\tan^2\theta=RS$

aa) $LS=\frac{\cot\theta}{\csc\theta-1}\cdot\frac{\csc\theta+1}{\csc\theta+1}=\frac{\cot\theta(\csc\theta+1)}{\csc^2\theta-1}=\frac{\cot\theta(\csc\theta+1)}{\cot^2\theta}=\frac{\csc\theta+1}{\cot\theta}=RS$

bb) $LS=\left(\tan\alpha+\tan\beta\right)\left(1-\cot\alpha\cot\beta\right)+\left(\cot\alpha+\cot\beta\right)\left(1-\tan\alpha\tan\beta\right)$

$=\left(\tan\alpha+\tan\beta\right)\left(1-\frac{1}{\tan\alpha\tan\beta}\right)+\left(\frac{1}{\tan\alpha}+\frac{1}{\tan\beta}\right)\left(1-\tan\alpha\tan\beta\right)$

$=\tan\alpha+\tan\beta-\frac{1}{\tan\beta}-\frac{1}{\tan\alpha}+\frac{1}{\tan\alpha}+\frac{1}{\tan\beta}-\tan\beta-\tan\alpha=0=RS$

cc) $LS=\left(\sin x-\tan x\right)\left(\cos x-\cot x\right)=\left(\sin x-\frac{\sin x}{\cos x}\right)\left(\cos x-\frac{\cos x}{\sin x}\right)$

$=\sin x\cos x-\cos x-\sin x+1$

$RS=\left(\cos x-1\right)\left(\sin x-1\right)=\sin x\cos x-\cos x-\sin x+1=LS$

dd) $LS=\frac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}\cdot\frac{1+\cos\theta+\sin\theta}{1+\cos\theta+\sin\theta}=\frac{1+\cos\theta+\sin\theta+\cos\theta+\cos^2\theta+\sin\theta\cos\theta+\sin\theta+\sin\theta\cos\theta+\sin^2\theta}{1+\cos\theta+\sin\theta+\cos\theta+\cos^2\theta+\sin\theta\cos\theta-\sin\theta-\sin\theta\cos\theta-\sin^2\theta}=\frac{2+2\cos\theta+2\sin\theta+2\sin\theta\cos\theta}{1+2\cos\theta+\cos^2\theta-\sin^2\theta}$

$=\frac{2(1+\cos\theta)+2\sin\theta(1+\cos\theta)}{1+2\cos\theta+\cos^2\theta+\cos^2-1}=\frac{2(1+\cos\theta)(1+\sin\theta)}{2\cos\theta(1+\cos\theta)}=\frac{1+\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\sec\theta+\tan\theta=RS$