MATHEMATICS 201-009-50
Precalculus
Martin Huard
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Semester Review
SOLUTIONS

1. Simplify each expression.

a) $\dfrac{6^2x^3y^{-3}}{12x^{-2}y}=\dfrac{36x^{3+2}}{12y^{1+3}}=\dfrac{3x^5}{y^4}$

b) $\left(\dfrac{4a^3b^{-1}c}{a^{-2}b^4}\right)^{-2}=\dfrac{4^{-2}a^{-6}b^2c^{-2}}{a^4b^{-8}}=\dfrac{b^{2+8}}{4^2a^{4+6}c^2}=\dfrac{b^{10}}{16a^{10}c^2}$

c) $\left(\dfrac{2x^2y^{-1}}{x^{-3}}\right)^3\left(\dfrac{x^{-9}y^6}{z^{12}}\right)^{-\frac{1}{3}}=\dfrac{2^3x^6y^{-3}}{x^{-9}}\cdot\dfrac{x^3y^{-2}}{z^{-4}}=\dfrac{8x^{6+3+9}z^4}{y^{2+3}}=\dfrac{8x^{18}z^4}{y^5}$

d) $\sqrt{\dfrac{18x^2}{y^8}}=\dfrac{\sqrt{18}\sqrt{x^2}}{\sqrt{y^8}}=\dfrac{3\sqrt{2}|x|}{y^4}$

e) $\sqrt{18x^8}-x^2\sqrt{8x^4}=3\sqrt{2}x^4-x^2\,2\sqrt{2}x^2=\sqrt{2}x^4$

f) $\sqrt[5]{t^{3n+1}}\sqrt[5]{t^{2n-1}}=\sqrt[5]{t^{3n+1+2n-1}}=\sqrt[5]{t^{5n}}=t$

g) $(3x-2)^2-(2x-5)=9x^2-12x+4-2x+5=9x^2-14x+9$

h)
$$\frac{1}{x-1}+\frac{1-x}{x^2+x+1}=\frac{x^2+x+1+(1-x)(x-1)}{(x-1)(x^2+x+1)}=\frac{x^2+x+1-x^2+2x-1}{(x-1)(x^2+x+1)}=\frac{3x}{(x-1)(x^2+x+1)}$$

i) $\dfrac{x^3-8}{x^2-5x+6}=\dfrac{(x-2)(x^2+2x+4)}{(x-3)(x-2)}=\dfrac{x^2+2x+4}{x-3}$

j)
$$\frac{3x}{x+2}-\frac{4x^2-5}{2x^2+3x-2}=\frac{3x}{x+2}-\frac{4x^2-5}{(2x-1)(x+2)}=\frac{3x(2x-1)-(4x^2-5)}{(2x-1)(x+2)}$$
$$=\frac{6x^2-3x-4x^2+5}{(2x-1)(x+2)}=\frac{2x^2-3x+5}{(2x-1)(x+2)}$$

k) $\dfrac{\dfrac{1}{2x-3}-\dfrac{1}{2x+3}}{\dfrac{1}{2x}-\dfrac{1}{2x+3}}=\dfrac{\frac{2x+3-(2x-3)}{(2x-3)(2x+3)}}{\frac{2x+3-2x}{2x(2x+3)}}=\dfrac{6}{(2x-3)(2x+3)}\dfrac{2x(2x+3)}{3}=\dfrac{4x}{2x-3}$

l) $\dfrac{x^2-4}{x^4-2x^2-8}\cdot\dfrac{x^2+2}{x^2}=\dfrac{(x^2-4)(x^2+2)}{x^2(x^2-4)(x^2+2)}=\dfrac{1}{x^2}$

m) $\dfrac{4x-6}{(x-1)^2} \div \dfrac{2x^2-3x}{x^2+2x-3} = \dfrac{2(2x-3)}{(x-1)^2} \cdot \dfrac{(x+3)(x-1)}{x(2x-3)} = \dfrac{2(x+3)}{x(x-1)}$

n) $\dfrac{5}{x} + \dfrac{1}{x+2} + \dfrac{3}{x^2+x-2} = \dfrac{5}{x} + \dfrac{1}{x+2} + \dfrac{3}{(x+2)(x-1)} = \dfrac{5(x+2)(x-1)+x(x-1)+3x}{x(x+2)(x-1)}$

$= \dfrac{5x^2+5x-10+x^2-x+3x}{x(x+2)(x-1)} = \dfrac{6x^2+7x-10}{x(x+2)(x-1)} = \dfrac{(6x-5)(x+2)}{x(x+2)(x-1)}$

$= \dfrac{6x-5}{x(x-1)}$

o) $\dfrac{(x+3)^{\frac{5}{3}} - 12x(x+3)^{\frac{-1}{3}}}{x^2-9} = \dfrac{(x+3)^{\frac{-1}{3}}\left((x+3)^2 - 12x\right)}{(x-3)(x+3)} = \dfrac{x^2+6x+9-12x}{(x-3)(x+3)^{\frac{4}{3}}} = \dfrac{x^2-6x+9}{(x-3)(x+3)^{\frac{4}{3}}}$

$= \dfrac{(x-3)^2}{(x-3)(x+3)^{\frac{4}{3}}} = \dfrac{x-3}{(x+3)^{\frac{4}{3}}}$

2. Rationalize the denominator.

a) $\dfrac{1}{2-\sqrt{3}} \cdot \dfrac{2+\sqrt{3}}{2+\sqrt{3}} = \dfrac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$

b) $\dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}} \cdot \dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}} = \dfrac{7+2\sqrt{5}\sqrt{7}+5}{7-5} = \dfrac{12+2\sqrt{35}}{2} = 6+\sqrt{35}$

3. Factor completely.

a) $x^2-12x+36 = (x-6)^2$

b) $8x^3+27 = (2x+3)(4x^2-6x+9)$

c) $12x^3+7x^2+x = x(12x^2+7x+1) = x(3x+1)(4x+1)$

d) $x(x+2)^{-\frac{1}{2}} + (x+2)^{\frac{1}{2}} = (x+2)^{-\frac{1}{2}}\left(x+(x+2)^1\right) = \dfrac{2x+2}{(x+2)^{\frac{1}{2}}} = \dfrac{2(x+1)}{(x+2)^{\frac{1}{2}}}$

e) $x^3+3x^2-4x-12 = x^2(x+3)-4(x+3) = (x+3)(x^2-4) = (x+3)(x-2)(x+2)$

f) $6(2x-1)^2(x+5)^5 + 2(2x-1)^3(x+5)^4 = 2(2x-1)^2(x+5)^4\left(3(x+5)+(2x-1)\right)$

$= 2(2x-1)^2(x+5)^4(5x+14)$

g) $8x^2(x-1)^{-3}(x+2)^{\frac{-3}{4}} - 4x(x-1)^{-4}(x+2)^{\frac{5}{4}} = 4x(x-1)^{-4}(x+2)^{\frac{-3}{4}}\left(2x(x-1)-(x+2)^2\right)$

$$= \frac{4x\left(2x^2-2x-x^2-4x-4\right)}{(x-1)^4(x+2)^{\frac{3}{4}}}$$

$$= \frac{4x\left(x^2-6x-4\right)}{(x-1)^4(x+2)^{\frac{3}{4}}}$$

4. Find all real solutions of each equation.

a) $\frac{4x-3}{6}+\frac{x}{4}=x-2$

$\frac{8x-6+3x}{12}=x-2$

$11x-6=12x-24$

$x=18$

b) $\frac{5}{x-2}=\frac{13}{2x-3}$

$5(2x-3)=13(x-2)$

$10x-15=13x-26$

$3x=11$

$x=\frac{11}{3}$

c) $x^2-13x+30=0$

$(x-10)(x-3)=0$

$x=3,10$

d) $x^2+6x-3=0$

$x^2+6x+9=3+9$

$(x+3)^2=12$

$x=-3\pm\sqrt{12}$

$=-3\pm 2\sqrt{3}$

e) $\frac{x}{x+2}-\frac{5}{x}=\frac{2x-2}{x^2+2x}$

$\frac{x^2-5(x+2)}{x(x+2)}=\frac{2x-2}{x(x+2)}$

$x^2-5x-10=2x-2$

$x^2-7x-8=0$

$(x-8)(x+1)=0$

$x=-1,8$

f) $(x+4)^{\frac{1}{2}}+5x(x+4)^{\frac{3}{2}}=0$

$(x+4)^{\frac{1}{2}}(1+5x(x+4))=0$

$(x+4)^{\frac{1}{2}}(5x^2+20x+1)=0$

$x=-4$ or $x=\frac{-20\pm\sqrt{20^2-4(5)(1)}}{2(5)}$

$=-2\pm\frac{\sqrt{380}}{10}=-2\pm\frac{\sqrt{95}}{5}$

g) $\sqrt{x-2}-8=0$

$\sqrt{x-2}=8$

$x-2=8^2$

$x=66$

h) $x^4-5x^2+6=0$

$(x^2-3)(x^2-2)=0$

$x=\pm\sqrt{3},\pm\sqrt{2}$

i) $5x^4-12x^3=0$

$x^3(5x-12)=0$

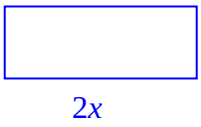
$x=0,\frac{12}{5}$

j) $|2x-5|=9$

$2x-5=9$ or $2x-5=-9$

$x=7$ $x=-2$

5. A rectangular garden is to be twice as long as it is wide. What should be its dimension if it is to have a total area of 80 m^2?



$A = bh$

$80 = x(2x)$

$40 = x^2$

$x = \cancel{-\sqrt{40}}, \sqrt{40}$

$x = 2\sqrt{10}$ meters

6. Solve each inequality. Give your answer in interval notation.

a) $9x - 8 \le 7x + 16$

$2x \le 24$

$x \le 12$

$x \in (-\infty, 12]$

b) $|2x - 3| < 5$

$-5 < 2x - 3 < 5$

$-2 < 2x < 8$

$-1 < x < 4$

$x \in (-1, 4)$

c) $x^2 - 2x - 3 \ge 0$

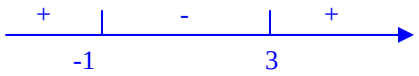
$(x-3)(x+1) \ge 0$

$(-\infty, -1)$ test number: $-3 \to 12$

$(-1, 3)$ test number: $0 \to -3$

$(3, \infty)$ test number: $4 \to 5$

Thus $(-\infty, -1] \cup [3, \infty)$



d) $\dfrac{2}{x+1} \le \dfrac{3}{x-1}$

$\dfrac{2}{x+1} - \dfrac{3}{x-1} \le 0$

$\dfrac{2(x-1) - 3(x+1)}{(x+1)(x-1)} \le 0$

$\dfrac{-x-5}{(x+1)(x-1)} \le 0$

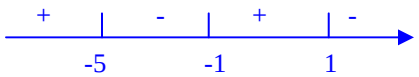
$(-\infty, -5)$ test number: $-6 \to \frac{1}{35}$

$(-5, -1)$ test number: $-2 \to -1$

$(-1, 1)$ test number: $0 \to 5$

$(1, \infty)$ test number: $1 \to \frac{-7}{3}$

Thus $[-5, -1) \cup (1, \infty)$



7. A car radiator contains 10 liters of a 30% antifreeze solution. How many liters will have to be replaced with pure antifreeze if the resulting solution is to be 50% antifreeze.

Let x be the number of liters of pure antifreeze.

$$(0.5)10 = (0.3)(10-x)+x$$
$$5 = 3+0.7x$$
$$x = \frac{2}{0.7} = \frac{20}{7} \text{ liters}$$

8. Let $A(-2,5)$ and $B(1,3)$ be points in the plane.

a) Find the length and midpoint of the segment AB.

$$d = \sqrt{(1-(-2))^2+(3-5)^2} = \sqrt{3^2+2^2} = \sqrt{13}$$
$$M = \left(\frac{-2+1}{2}, \frac{5+3}{2}\right) = \left(\frac{-1}{2}, 4\right)$$

b) Find an equation for the line passing through the points A and B.

$m = \frac{3-5}{1-(-2)} = \frac{-2}{3}$ $\quad y = mx+b$ $\quad 5 = \frac{-2}{3}(-2)+b$ $\quad b = \frac{11}{3}$ $\quad$ Thus $y = \frac{-2}{3}x+\frac{11}{3}$

c) Find an equation for the line perpendicular to the segment AB and passing through the midpoint of the segment AB.

$m = \frac{3}{2}$ $\quad y = mx+b$ $\quad 4 = \frac{3}{2}\left(\frac{-1}{2}\right)+b$ $\quad b = \frac{19}{3}$ $\quad$ Thus $y = \frac{3}{2}x+\frac{19}{4}$

d) Find the equation for the circle having A and B as the endpoints of a diameter.

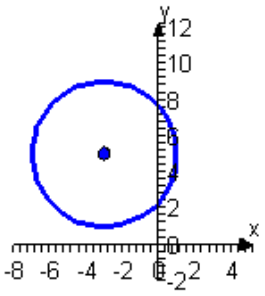
Radius: $\frac{\sqrt{13}}{2}$ $\quad \left(x+\frac{1}{2}\right)^2+(y-4)^2 = \frac{13}{4}$

Center: $\left(\frac{-1}{2}, 4\right)$

9. Find the radius and center for the circle $x^2+6x+y^2-10y+18=0$ and sketch the graph.

$$x^2+6x+y^2-10y+18=0$$
$$x^2+6x+9+y^2-10y+25 = -18+9+25$$
$$(x+3)^2+(y-5)^2 = 16$$

Radius: 4 $\quad$ Center: $(-3,5)$



10. Find the domain of the following functions and determine whether they are even or odd.

a) $f(x) = \dfrac{x}{x^2-4}$ $\quad x^2-4=0$ $\quad x = \pm 2$ $\quad$ Domain: $\mathbb{R}/\{-2,2\}$

$$f(-x) = \frac{-x}{(-x)^2-4} = \frac{-x}{x^2-4} = -f(x)$$ hence f is odd

b) $f(x)=\dfrac{\sqrt{x-2}+5}{x-10}$ $x-2\ge 0$ $x-10=0$ Domain: $[2,10)\cup(10,\infty)$

$x\ge 2$ $x=10$

$f(-x)=\dfrac{\sqrt{-x-2}+5}{-x-10}\neq \pm f(x)$ thus f in neither odd nor even

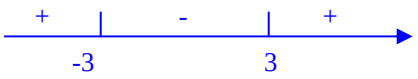
c) $f(x)=\sqrt{x^2-9}$ $x^2-9\ge 0$

$(x-3)(x+3)\ge 0$

$(-\infty,-3)$ test number: $-4\to 7$

$(-3,3)$ test number: $0\to -9$

$(3,\infty)$ test number: $4\to 7$



Thus the domain is $(-\infty,-3]\cup[3,\infty)$

$f(-x)=\sqrt{(-x)^2-9}=\sqrt{x^2-9}=f(x)$ thus f is even

d) $f(x)=\sqrt{6+x-x^2}$ $6+x-x^2\ge 0$

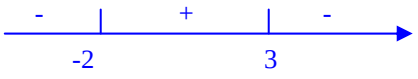
$-(x^2-x-6)\ge 0$

$-(x-3)(x+2)\ge 0$

$(-\infty,-2)$ test number: $-3\to -6$

$(-2,3)$ test number: $0\to 6$

$(3,\infty)$ test number: $4\to -6$



Thus the domain is $[-2,3]$

$f(-x)=\sqrt{6-x-(-x^2)}=\sqrt{6-x-x^2}\neq \pm f(x)$ thus f in neither odd nor even

11. Let $f(x)=x^2+x-1$. Find

a) $f(2t+1)=(2t+1)^2+(2t+1)-1=4t^2+4t+1+2t+1-1=4t^2+6t+1$

b) $f\left(\frac{x}{x+1}\right)=\left(\dfrac{x}{x+1}\right)^2+\dfrac{x}{x+1}-1=\dfrac{x^2+x(x+1)-(x+1)^2}{(x+1)^2}=\dfrac{x^2+x^2+x-x^2-2x-1}{(x+1)^2}$

$=\dfrac{x^2-x-1}{(x+1)^2}$

c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{(x+h)^2+(x+h)-1-(x^2+x-1)}{h}$

$=\dfrac{x^2+2xh+h^2+x+h-1-x^2-x+1}{h}$

$=\dfrac{h(2x+h+1)}{h}=2x+h+1$

12. Let $f(x)=\dfrac{x+2}{x+4}$. Find

a) $f(x^2-1)=\dfrac{x^2-1+2}{x^2-1+4}=\dfrac{x^2+1}{x^2+3}$

b) $f\left(\frac{2}{x+3}\right)=\dfrac{\frac{2}{x+3}+2}{\frac{2}{x+3}+4}=\dfrac{\frac{2+2(x+3)}{x+3}}{\frac{2+4(x+3)}{x+3}}=\dfrac{2x+8}{x+3}\dfrac{x+3}{4x+14}=\dfrac{2(x+4)}{2(2x+7)}=\dfrac{x+4}{2x+7}$

c) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{\frac{x+h+2}{x+h+4}-\frac{x+2}{x+4}}{h}=\dfrac{(x+h+2)(x+4)-(x+2)(x+h+4)}{(x+h+4)(x+4)}\dfrac{1}{h}$

$=\dfrac{x^2+hx+2x+4x+4h+8-x^2-xh-4x-2x-2h-8}{h(x+h+4)(x+4)}$

$=\dfrac{2h}{h(x+h+4)(x+4)}=\dfrac{2}{(x+h+4)(x+4)}$

13. Find the x and y intercepts of the given function and sketch the graph.

a) $f(x)=\sqrt{x+1}-2$

x-int: $\sqrt{x+1}-2=0$

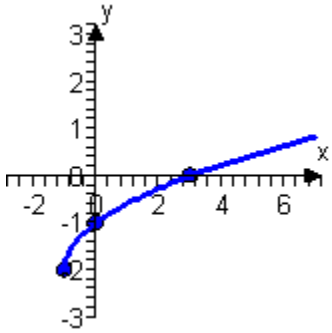
$\sqrt{x+1}=2$

$x+1=4$

$x=3$

y-int: $f(0)=\sqrt{0+1}-2$

$=-1$



b) $f(x)=x^3+8$

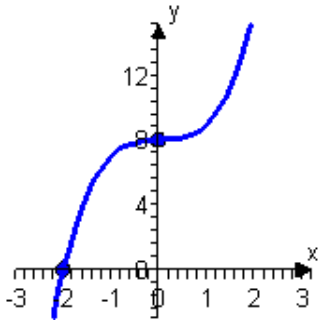
x-int: $x^3+8=0$

$x^3=-8$

$x=\sqrt[3]{-8}$

$=-2$

y-int: $f(0)=8$



c) $f(x)=3|x-2|+5$

x-int: $3|x-2|+5=0$

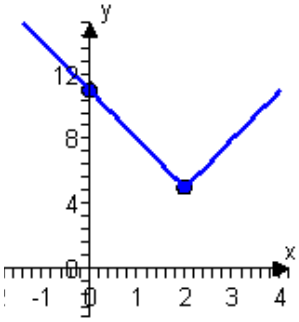
$3|x-2|=-5$

$|x-2|=\frac{-5}{3}$

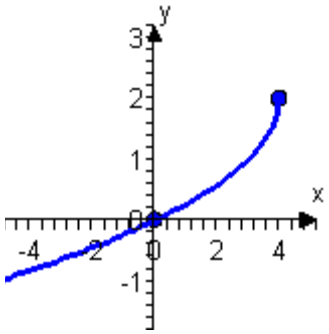
No solutions

y-int: $f(0)=3|0-2|+5$

$=11$



d) $f(x) = -\sqrt{4-x} + 2$



x-int: $-\sqrt{4-x} + 2 = 0$

$\sqrt{4-x} = 2$

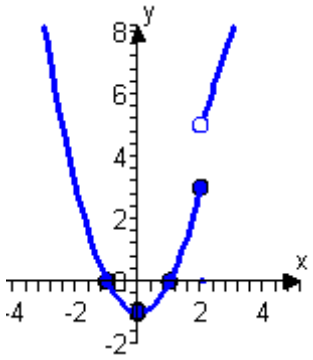
$4 - x = 4$

$x = 0$

y-int: $f(0) = -\sqrt{4} + 2$

$= 0$

e) $f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 2 \\ 3x - 1 & \text{if } x > 2 \end{cases}$



x-int: $x^2 - 1 = 0$ $\qquad 3x - 1 = 0$

$x^2 = 1$ $\qquad x = \frac{1}{3} \not> 2$

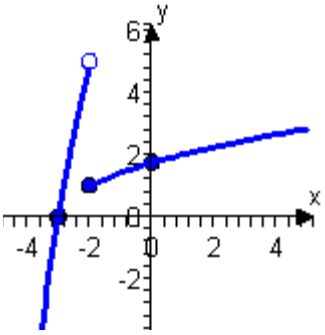
$x = \pm 1 < 2$

Hence the x-ints are $x = \pm 1$

y-int: $f(0) = 0^2 - 1$

$= -1$

f) $f(x) = \begin{cases} 9 - x^2 & \text{if } x < -2 \\ \sqrt{x+3} & \text{if } x \ge -2 \end{cases}$



x-int: $9 - x^2 = 0$ $\qquad \sqrt{x+3} = 0$

$x^2 = 9$ $\qquad x = -3 \not> -2$

$x = -3 < -2$

or $x = 3 \not< -2$

Hence the x-int. is $x = -3$ are $x = \pm 1$

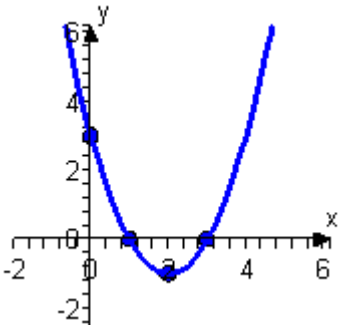
y-int: $f(0) = \sqrt{0+3}$

$= \sqrt{3}$

14. For the given function,

i) find the vertex;
ii) find the intercepts;
iii) find maximum or minimum value that *f* takes;
iv) find the domain and range;
v) sketch the graph.

a) $f(x) = x^2 - 4x + 3$



(i) $f(x) = x^2 - 4x + 3$ Vertex: $(2,-1)$

$= x^2 - 4x + 4 + 3 - 4$

$= (x-2)^2 - 1$

(ii) *x*-int: $x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 1, 3$

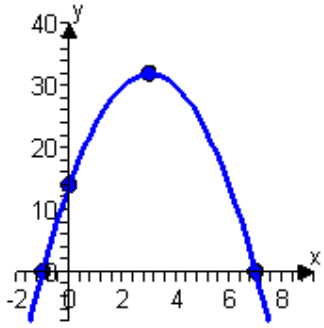
y-int: $f(0) = 3$

(iii) Minimum value of -1 when $x = 2$

(iv) Domain: $\mathbb{R}$

Range: $[-1, \infty)$

b) $f(x) = -2x^2 + 12x + 14$



(i) $f(x) = -2x^2 + 12x + 14$

$= -2(x^2 - 6x + 9) + 14 + 2(9)$

$= -2(x-3)^2 + 32$

Vertex: $(3, 32)$

(ii) *x*-int: $-2x^2 + 12x + 14 = 0$

$x^2 - 6x - 7 = 0$

$(x-7)(x+1) = 0$

$x = -1, 7$

y-int: $f(0) = 14$

(iii) Maximum value of 32 when $x = 3$

(iv) Domain: $\mathbb{R}$

Range: $(-\infty, 32]$

15. A sports center has 25 skidoos to rent. The owner finds that if he charges \$60 per day, all of his skidoos will be rented. However, for each \$4 increase in the price, he will rent one skidoo less. What price should he charge to maximize his revenue?

Let x be the number of \$4 increase in the price.

$$\begin{aligned} P &= (\#\text{ skidoos rented}) \times (\text{price}) \\ &= (25-x)(60+4x) \\ &= -4x^2 + 40x + 1500 \\ &= -4(x^2 - 10x + 25) + 1500 + 4(25) \\ &= -4(x-5)^2 + 1600 \end{aligned}$$

The profit will be a maximum of \$1600 when $x = 4$, that is, when the price is \$80 per day.

16. Let $f(x) = x^2 + x - 2$ and $g(x) = \sqrt{x} + 1$. Find the following function, and state the domain of each.

a) $(f+g)(x) = f(x) + g(x) = x^2 + x - 2 + \sqrt{x} + 1 = x^2 + x + \sqrt{x} - 1$

b) $(f \cdot g)(x) = f(x)g(x) = (x^2 + x - 2)(\sqrt{x} + 1) = x^{\frac{5}{2}} + x^2 + x^{\frac{3}{2}} + x - 2\sqrt{x} - 2$

c) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x} + 1) = (\sqrt{x} + 1)^2 + (\sqrt{x} + 1) - 2 = x + 2\sqrt{x} + 1 + \sqrt{x} + 1 - 2$

$= x + 3\sqrt{x} - 1$

d) $(g \circ f)(x) = g(f(x)) = g(x^2 + x - 2) = \sqrt{x^2 + x - 2} + 1$

e) $(f \circ f)(x) = f(f(x)) = f(x^2 + x - 2) = (x^2 + x - 2)^2 + (x^2 + x - 2) - 2$

$= x^4 + x^3 - 2x^2 + x^3 + x^2 - 2x - 2x^2 - 2x + 4 + x^2 + x - 2 - 2$

$= x^4 + 2x^3 - 2x^2 - 3x$

f) $(g \circ g)(x) = g(g(x)) = g(\sqrt{x} + 1) = \sqrt{\sqrt{x} + 1} + 1$

17. For the given function,

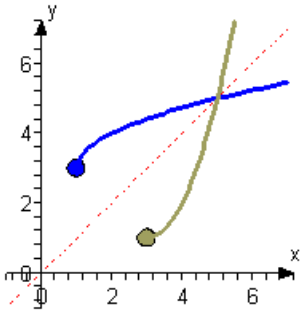
i) find the inverse f^{-1} of f;

ii) find the domain and range of f and f^{-1};

iii) verify that $\left(f^{-1}\circ f\right)(x)=x$ and $\left(f\circ f^{-1}\right)(x)=x$;

iv) sketch the graph of f and f^{-1}.

a) $f(x)=\sqrt{x-1}+3$



(i) $y=\sqrt{x-1}+3$ $\quad f^{-1}(x)=(x-3)^2+1$

$x=\sqrt{y-1}+3$

$(x-3)^2=y-1$

$y=(x-3)^2+1$

(ii) Domain of f: $[1,\infty)$ Range of f: $[3,\infty)$

Domain of f^{-1}: $[3,\infty)$ Range of f^{-1}: $[1,\infty)$

(iii) $\left(f^{-1}\circ f\right)(x)=f^{-1}\left(f(x)\right)$

$=f^{-1}\left(\sqrt{x-1}+3\right)$

$=\left(\sqrt{x-1}+3-3\right)^2+1$

$=\left(\sqrt{x-1}\right)^2+1$

$=x-1+1$

$=x$

$\left(f\circ f^{-1}\right)(x)=f\left(f^{-1}(x)\right)$

$=f\left((x-3)^2+1\right)$

$=\sqrt{(x-3)^2+1-1}+3$

$=\sqrt{(x-3)^2}+3$

$=|x-3|+3$

$=x-3+3$ if $x\ge 3$

$=x$

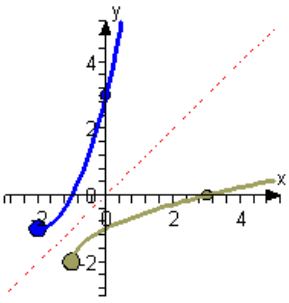
b) $f(x)=(x+2)^2-1 \qquad x\ge -2$

(i) $y=(x+2)^2-1$ $\quad f^{-1}(x)=\sqrt{x+1}-2$

$x=(y+2)^2-1$

$x+1=(y+2)^2$

$y=\sqrt{x+1}-2$



(ii) Domain of f: $[-2,\infty)$ Range of f: $[-1,\infty)$

Domain of f^{-1}: $[-1,\infty)$ Range of f^{-1}: $[-2,\infty)$

(iii) $\left(f^{-1}\circ f\right)(x)=f^{-1}\left(f(x)\right)$

$=f^{-1}\left((x+2)^2-1\right)$

$=\sqrt{(x+2)^2-1+1}-2$

$=\sqrt{(x+2)^2}-2$

$=|x+2|-2$

$=x+2-2 \qquad \text{if } x\geq -2$

$=x$

$\left(f\circ f^{-1}\right)(x)=f\left(f^{-1}(x)\right)$

$=f\left(\sqrt{x+1}-2\right)$

$=\left(\sqrt{x+1}-2+2\right)^2-1$

$=\left(\sqrt{x+1}\right)^2-1$

$=x+1-1$

$=x$

18. Assuming that f is invertible, find $f^{-1}(2)$ if $f(3)=2$.

$f(3)=2 \quad \Leftrightarrow \quad 3=f^{-1}(2)$

19. Divide the following.

a) $\dfrac{x^4-3x^2+x-5}{x^2+2x}$

$$\begin{array}{r} x^2-2x+1 \\ x^2+2x\overline{)\,x^4\quad -3x^2+x-5} \\ \underline{x^4+2x^3} \\ -2x^3-3x^2+x-5 \\ \underline{-2x^3-4x^2} \\ x^2+x-5 \\ \underline{x^2+2x} \\ -x-5 \end{array}$$

Thus $\dfrac{x^4-3x^2+x-5}{x^2+2x}=x^2-2x+1-\dfrac{x+5}{x^2+2x}$

b) $\dfrac{x^6-3x^5+x^2-2}{x^3+3x+1}$

$$
\begin{array}{r}
x^3-3x^2-3x+8 \\
x^3+3x+1\overline{)\,x^6-3x^5\qquad\qquad+x^2\quad-2} \\
\underline{x^6\qquad+3x^4+x^3\qquad\qquad\qquad\quad} \\
-3x^5-3x^4-\;x^3+x^2\qquad-2 \\
\underline{-3x^5\qquad-9x^3-3x^2\qquad\quad} \\
-3x^4+8x^3+4x^2\qquad-2 \\
\underline{-3x^4\qquad-9x^2-3x\quad} \\
8x^3+13x^2+3x-2 \\
\underline{8x^3\qquad+24x+8} \\
13x^2-21x-10
\end{array}
$$

Thus $\dfrac{x^6-3x^5+x^2-2}{x^3+3x+1}=x^3-3x^2-3x+8+\dfrac{13x^2-21x-10}{x^3+3x+1}$

c) $\dfrac{x^8-1}{x^7+1}$

$$
\begin{array}{r}
x \\
x^7+1\overline{)\,x^8\quad-1} \\
\underline{x^8+x\quad} \\
-x-1
\end{array}
$$

Thus $\dfrac{x^8-1}{x^7+1}=x-\dfrac{x+1}{x^7+1}$

20. Factor the following polynomials.

a) $p(x)=x^3-4x^2+x+6$

$a_0=6\rightarrow\pm1,\pm2,\pm3,\pm6$

$a_3=1\rightarrow\pm1$

Possible zeros: $\pm1,\pm2,\pm3,\pm6$

$p(1)=4$

$p(-1)=0$

$$\begin{array}{r} x^2-5x+6 \\ x+1\overline{)\,x^3-4x^2+x+6} \\ \underline{x^3+x^2} \\ -5x^2+x+6 \\ \underline{-5x^2-5x} \\ 6x+6 \\ \underline{6x+6} \\ 0 \end{array}$$

Thus $p(x)=x^3-4x^2+x+6$

$=(x+1)(x^2-5x+6)$

$=(x+1)(x-2)(x-3)$

b) $p(x)=x^3-4x^2-11x+30$

$a_0=30\rightarrow\pm1,\pm2,\pm3,\pm5,\pm6,\pm10,\pm15,\pm30$

$a_3=1\rightarrow\pm1$

Possible zeros: $\pm1,\pm2,\pm3,\pm5,\pm6,\pm10,\pm15,\pm30$

$p(1)=16$

$p(2)=0$

$$\begin{array}{r} x^2-2x-15 \\ x-2\overline{)\,x^3-4x^2-11x+30} \\ \underline{x^3-2x^2} \\ -2x^2-11x+30 \\ \underline{-2x^2+4x} \\ -15x+30 \\ \underline{-15x+30} \\ 0 \end{array}$$

Thus $p(x)=x^3-4x^2+x+6$

$=(x-2)(x^2-2x-15)$

$=(x-2)(x+3)(x-5)$

c) $p(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$

$a_0 = 36 \to \pm1, \pm2, \pm3, \pm6, \pm9, \pm12, \pm18, \pm36$

$a_4 = 1 \to \pm1$

Possible zeros: $\pm1, \pm2, \pm3, \pm6, \pm9, \pm12, \pm18, \pm36$

$p(1) = 36$

$p(-1) = 16$

$p(2) = 16$

$p(-2) = 0$

$$\begin{array}{r} x^3 - 4x^2 - 3x + 18 \\ x+2\overline{)\,x^4 - 2x^3 - 11x^2 + 12x + 36} \\ \underline{x^4 + 2x^3} \\ -4x^3 - 11x^2 + 12x + 36 \\ \underline{-4x^3 - 8x^2} \\ -3x^2 + 12x + 36 \\ \underline{-3x^2 - 6x} \\ 18x + 36 \\ \underline{18x + 36} \\ 0 \end{array}$$

Thus $p(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$

$= (x+2)(x^3 - 4x^2 - 3x + 18)$

$a_0 = 18 \to \pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

$a_4 = 1 \to \pm1$

Possible zeros: $\pm1, \pm2, \pm3, \pm6, \pm12, \pm18, \pm36$

$p(-2) = 0$

$$\begin{array}{r} x^2 - 6x + 9 \\ x+2\overline{)\,x^3 - 4x^2 - 3x + 18} \\ \underline{x^4 + 2x^2} \\ -6x^2 - 3x + 18 \\ \underline{-6x^2 - 12x} \\ 9x + 18 \\ \underline{9x + 18} \\ 0 \end{array}$$

Ergo $p(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$

$= (x+2)(x^3 - 4x^2 - 3x + 18)$

$= (x+2)^2(x^2 - 6x + 9)$

$= (x+2)^2(x-3)^2$

d) $p(x)=2x^3+3x^2-23x-12$

$a_0=12\rightarrow\pm1,\pm2,\pm3,\pm4,\pm6,\pm12$

$a_3=1\rightarrow\pm1,\pm2$

Possible zeros: $\pm\frac{1}{2},\pm1,\pm\frac{3}{2},\pm2,\pm3,\pm4,\pm6,\pm12$

$p\left(\frac{1}{2}\right)=\frac{-45}{2}$

$p\left(\frac{-1}{2}\right)=0$

$$\begin{array}{r} 2x^2+2x-24 \\ x+\frac{1}{2}\overline{)\,2x^3+3x^2-23x-12} \\ \underline{2x^3+x^2} \\ 2x^2-23x-12 \\ \underline{2x^2+x} \\ -24x-12 \\ \underline{-24x-12} \\ 0 \end{array}$$

Thus $p(x)=2x^3+3x^2-23x-12$

$=\left(x+\frac{1}{2}\right)\left(2x^2+2x-24\right)$

$=(2x+1)\left(x^2+x-12\right)$

$=(2x+1)(x+4)(x-3)$

21. Solve the following equations. (*Hint*: find all rational zeros.)

a) $x^3-5x^2-4x+20=0$

$a_0=20\rightarrow\pm1,\pm2,\pm4,\pm5,\pm10,\pm20$

$a_3=1\rightarrow\pm1$

Possible zeros: $\pm1,\pm2,\pm4,\pm5,\pm10,\pm20$

$p(1)=12$

$p(-1)=18$

$p(2)=0$

$$\begin{array}{r} x^2-3x-10 \\ x-2\overline{)\,x^3-5x^2-4x+20} \\ \underline{x^3-2x^2} \\ -3x^2-4x+20 \\ \underline{-3x^2+6x} \\ -10x+20 \\ \underline{-10x+20} \\ 0 \end{array}$$

Thus $x^3-5x^2-4x+20=0$

$(x-2)\left(x^2-3x-10\right)=0$

$(x-2)(x-5)(x+2)=0$

$x=-2,2,5$

b) $x^3+2x^2-2x-3=0$

$a_0=3\rightarrow\pm1,\pm3$

$a_3=1\rightarrow\pm1$

Possible zeros: $\pm1,\pm3$

$p(1)=-2$

$p(-1)=0$

$$\begin{array}{r} x^2+x-3 \\ x+1\overline{)\,x^3+2x^2-2x-3} \\ \underline{x^3+x^2} \\ x^2-2x-3 \\ \underline{x^2+x} \\ -3x-3 \\ \underline{-3x-3} \\ 0 \end{array}$$

Thus $x^3+2x^2-2x-3=0$

$(x+1)(x^2+x-3)=0$

$x=-1$ or $x^2+x-3=0$

$$x=\frac{-1\pm\sqrt{1^2-4(1)(-3)}}{2}$$

$$=\tfrac{-1}{2}\pm\tfrac{\sqrt{13}}{2}$$

c) $2x^3+3x^2-8x+3=0$

$a_0=3\rightarrow\pm1,\pm3$

$a_3=1\rightarrow\pm1,\pm2$

Possible zeros: $\pm\frac{1}{2},\pm1,\pm\frac{3}{2},\pm3$

$p(1)=0$

$$\begin{array}{r} 2x^2+5x-3 \\ x-1\overline{)\,2x^3+3x^2-8x+3} \\ \underline{2x^3-2x^2} \\ 5x^2-8x+3 \\ \underline{5x^2-5x} \\ -3x+3 \\ \underline{-3x+3} \\ 0 \end{array}$$

Thus $2x^3+3x^2-8x+3=0$

$(x+1)(2x^2+5x-3)=0$

$(x+1)(2x-1)(x+3)=0$

$x=-3,-1,\frac{1}{2}$

d) $4x^3 - 7x + 3 = 0$

$a_0 = 36 \to \pm 1, \pm 3$

$a_3 = 1 \to \pm 1, \pm 2, \pm 4$

Possible zeros: $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{3}{2}, \pm 3$

$p(1) = 0$

$$\begin{array}{r} 4x^2 + 4x - 3 \\ x-1 \overline{)\,4x^3 \quad -7x+3} \\ \underline{4x^4 - 4x^2} \\ 4x^2 - 7x + 3 \\ \underline{4x^2 - 4x} \\ -3x + 3 \\ \underline{-3x + 3} \\ 0 \end{array}$$

Thus $4x^3 - 7x + 3 = 0$

$(x-1)(4x^2 + 4x - 3) = 0$

$(x-1)(2x+3)(2x-1) = 0$

$x = \frac{-3}{2}, \frac{1}{2}, 1$

22. Find the domain, range, intercepts and asymptotes (if any) for the following functions, and sketch the graph.

a) $f(x) = 3^x - 9$

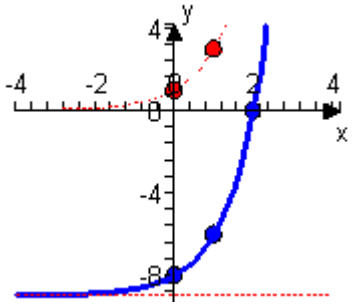
Domain: $\mathbb{R}$ Range: $(-9, \infty)$

x-int: $3^x - 9 = 0$ y-int: $f(0) = -8$

$3^x = 9$

$x = \log_3 9 = 2$

Horizontal Asymptote: $y = -9$



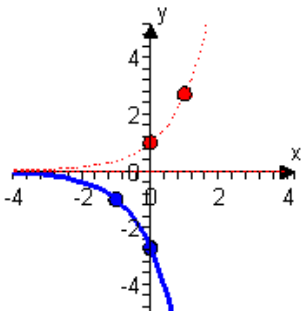
b) $f(x) = -e^{x+1}$

Domain: $\mathbb{R}$ Range: $(-\infty, 0)$

x-int: $-e^{x+1} = 0$ y-int: $f(0) = -e$

No solution

Horizontal Asymptote: $y = 0$



c) $f(x) = -\left(\frac{1}{2}\right)^{x+2} + 4$

Domain: $\mathbb{R}$ Range: $(-\infty, 4)$

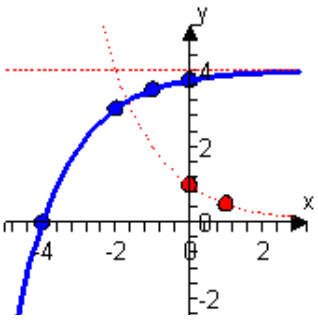
x-int: $-\left(\frac{1}{2}\right)^{x+2} + 4 = 0$ y-int: $f(0) = \frac{15}{4}$

$\left(\frac{1}{2}\right)^{x+2} = 4$

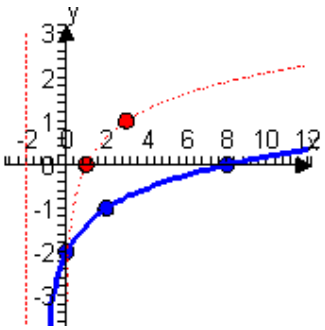
$x + 2 = \log_{\frac{1}{2}} 4$

$x = -4$

Horizontal Asymptote: $y = 4$



d) $f(x)=\log_3(x+1)-2$



Domain: $(-1,\infty)$ Range: $\mathbb{R}$

x-int: $\log_3(x+1)-2=0$ y-int: $f(0)=-2$

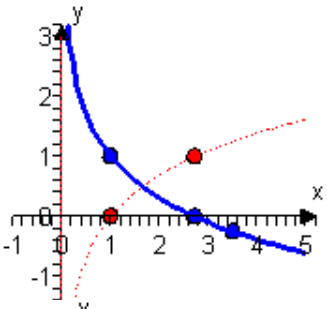
$\log_3(x+1)=2$

$x+1=3^2$

$x=8$

Vertical Asymptote: $x=-1$

e) $f(x)=-\ln(x)+1$



Domain: $(0,\infty)$ Range: $\mathbb{R}$

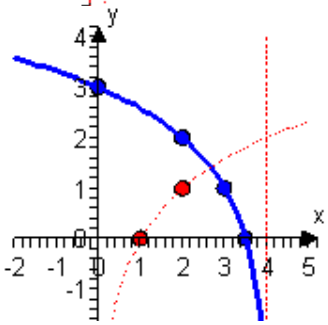
x-int: $-\ln x+1=0$ y-int: $f(0)\not\exists$

$\ln x=1$

$x=e^1=e$

Vertical Asymptote: $x=0$

f) $f(x)=\log_2(4-x)+1$



Domain: $(-\infty,4)$ Range: $\mathbb{R}$

x-int: $\log_2(4-x)+1=0$ y-int: $f(0)=3$

$\log_2(4-x)=-1$

$4-x=2^{-1}$

$x=\frac{7}{2}$

Vertical Asymptote: $x=4$

23. Write as a single logarithm.

a) $2\log_3 x-5\log_3 y^2=\log_3 x^2-\log_3 y^{10}=\log_3\dfrac{x^2}{y^{10}}$

b) $5\log_2 x^{-2}-3\log_2(x+1)+1=\log_2 x^{-10}-\log_2(x+1)^3+\log_2 2=\log_2\dfrac{2}{x^{10}(x+1)^3}$

24. Write the expression as a sum, difference, and/or multiple of logarithms.

a) $\log_3\frac{3x^5}{y^2}=\log_3 3+\log_3 x^5-\log_3 y^2=1+5\log_3 x-2\log_3 y$

b) $\ln\frac{x-1}{x^3}=\ln(x-1)-\ln x^3=\ln(x-1)-3\ln x$

25. Find the solution of the equation.

a) $3e^{-5x} = 132$

$e^{-5x} = 44$

$-5x = \ln 44$

$x = \frac{-1}{5}\ln 4$

b) $3^{2x+1} - 4 = 7$

$3^{2x+1} = 11$

$2x+1 = \log_3 11$

$x = \frac{1}{2}\log_3 11 - \frac{1}{2}$

c) $e^{2x} - 7e^x + 10 = 0$

$(e^x - 5)(e^x - 2) = 0$

$e^x = 5 \quad \text{or} \quad e^x = 2$

$x = \ln 5, \ln 2$

d) $2\ln 4x = 15$

$\ln 4x = \frac{15}{2}$

$4x = e^{\frac{15}{2}}$

$x = \frac{1}{4}e^{\frac{15}{2}}$

e) $4\log_3(x+5) - 2 = 7$

$4\log_3(x+5) = 9$

$\log_3(x+5) = \frac{9}{4}$

$x+5 = 3^{\frac{9}{4}}$

$x = 3^{\frac{9}{4}} - 5$

f) $\log x + \log(x+1) = \log 12$

$\log(x(x+1)) = \log 12$

$x^2 + x = 12$

$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$x = \cancel{-4}, 3$

26. A culture contains 10 000 bacteria initially. After an hour the bacteria count is 25 000.

a) Find the population after 3 hours.

$P = P_o e^{kt}$

$P(t) = 10000e^{kt}$

$P(1) = 10000e^t = 25000$

$e^t = \frac{25000}{10000}$

$t = \ln\frac{5}{2}$

Hence $P(t) = 10000e^{t\ln\frac{5}{2}}$

Ergo, $P(3) = 10000e^{3\ln\frac{5}{2}} = 19531250$

b) How long will it take for the population to double?

$20000 = 10000e^{t\ln\frac{5}{2}}$

$2 = e^{t\ln\frac{5}{2}}$

$t\ln\frac{5}{2} = \ln 2$

$t = \frac{\ln 2}{\ln\frac{5}{2}} \approx 0.274$

Hence it will take 0.274 hours or 16.5 minutes

27. A man invests \$5000 in a mutual fund which pays 8% per year, compounded monthly.

a) How much money will the man have in 5 years?

$$A = P\left(1+\tfrac{r}{n}\right)^{nt}$$

$$A(t) = 5000\left(1+\tfrac{0.08}{12}\right)^{12t}$$

$$A(5) = 5000\left(1+\tfrac{0.08}{12}\right)^{60} \simeq \$7449.23$$

b) How long would it take for the amount to triple?

$$15000 = 5000\left(1+\tfrac{0.08}{12}\right)^{12t}$$

$$3 = \left(1+\tfrac{0.08}{12}\right)^{12t}$$

$$12t = \ln_{1+\frac{0.08}{12}} 3$$

$$t = \tfrac{1}{12}\ln_{1+\frac{0.08}{12}} 3 \simeq 13.8 \text{ years}$$

28. Find the central angle θ in a circle of radius 4m that subtends an arc of length 5π m. Give the answer in degrees and in radians.

$$s = r\theta$$

$$5\pi = 4\theta$$

$$\theta = \tfrac{5\pi}{4}\text{ rad} \quad \text{or} \quad \tfrac{5\pi}{4}\tfrac{180^\circ}{\pi} = 135^\circ$$

29. Find the area of a sector with central angle 150^o in a circle of radius 4m.

$$150^\circ \tfrac{\pi}{180^\circ} = \tfrac{5\pi}{6}\text{ rad}$$

$$A = \tfrac{1}{2}r^2\theta$$

$$= \tfrac{1}{2}4^2\tfrac{5\pi}{6}$$

$$= \tfrac{20\pi}{3}\text{m}^2$$

30. Find the ***exact value*** of the following expressions.

a) $\cos\tfrac{2\pi}{3} = \tfrac{-1}{2}$

b) $\sec\tfrac{3\pi}{4} = \tfrac{1}{\cos\frac{3\pi}{4}} = \tfrac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$

c) $\cot\tfrac{13\pi}{3} = \cot\left(\tfrac{13\pi}{3} - 4\pi\right) = \cot\tfrac{\pi}{3} = \tfrac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \tfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3}$

d) $\tan\tfrac{\pi}{12} = \tan\left(\tfrac{\pi}{4} - \tfrac{\pi}{6}\right) = \dfrac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{5}} = \dfrac{\frac{\sqrt{2}}{2} - \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}}{1+\frac{\sqrt{2}}{2}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}} = \dfrac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}\cdot\dfrac{\sqrt{3}-1}{\sqrt{3}-1} = \dfrac{3+1-2\sqrt{3}}{2}$

$$= 2-\sqrt{3}$$

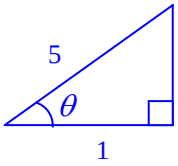
e) $\sin\tfrac{7\pi}{12} = \sin\left(\tfrac{\pi}{4}+\tfrac{\pi}{3}\right) = \sin\tfrac{\pi}{4}\cos\tfrac{\pi}{3} + \sin\tfrac{\pi}{3}\cos\tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2}\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\tfrac{\sqrt{2}}{2} = \tfrac{\sqrt{2}+\sqrt{6}}{4}$

f) $2\sin\tfrac{\pi}{12}\cos\tfrac{\pi}{12} = \sin\left(2\tfrac{\pi}{12}\right) = \sin\tfrac{\pi}{6} = \tfrac{1}{2}$

g) $\cos^2\tfrac{\pi}{8} - \sin^2\tfrac{\pi}{8} = \cos\left(2\tfrac{\pi}{8}\right) = \cos\tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2}$

h) $\sin\left(\arccos\tfrac{1}{5}\right) = \sin\theta$ $\qquad \arccos\tfrac{1}{5} = \theta$

$= \tfrac{2\sqrt{6}}{5}$ $\qquad \cos\theta = \tfrac{1}{5}$



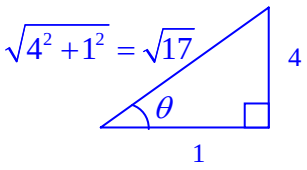
$$\sqrt{5^2-1^2} = \sqrt{24} = 2\sqrt{6}$$

i) $\sec(\arctan 4) = \sec\theta$ $\qquad \arctan 4 = \theta$

$= \dfrac{1}{\cos\theta}$ $\qquad \tan\theta = 4$

$= \dfrac{1}{\frac{1}{\sqrt{17}}}$

$= \sqrt{17}$



j) $\csc\left(\arcsin\frac{3}{7}\right) = \csc\theta$ $\qquad \arcsin\frac{3}{7} = \theta$

$= \frac{7}{3}$ $\qquad \sin\theta = \frac{3}{7}$

$\csc\theta = \frac{7}{3}$

k) $\arccos\left(\cos\frac{7\pi}{4}\right) = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

l) $\arctan\left(\tan\frac{5\pi}{4}\right) = \arctan\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$

m) $\sin\left(\arcsin\frac{2}{3} + \arccos\frac{1}{3}\right) = \sin\left(\arcsin\frac{2}{3}\right)\cos\left(\arccos\frac{1}{3}\right) + \sin\left(\arccos\frac{1}{3}\right)\cos\left(\arcsin\frac{2}{3}\right)$

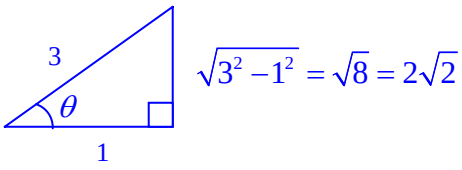
$= \frac{2}{3}\frac{1}{3} + \sin\theta\cos\phi$

$= \frac{2}{3}\frac{1}{3} + \frac{2\sqrt{2}}{3}\frac{\sqrt{5}}{3}$

$= \frac{2+2\sqrt{10}}{9}$

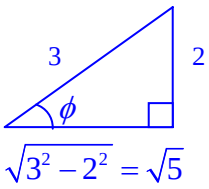
$\arccos\frac{1}{3} = \theta$

$\cos\theta = \frac{1}{3}$



$\arcsin\frac{2}{3} = \phi$

$\sin\phi = \frac{2}{3}$

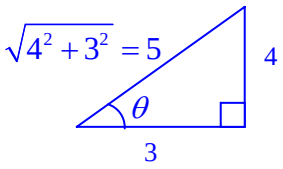


n) $\cos\left(2\arctan\frac{4}{3}\right) = \cos^2\left(\arctan\frac{4}{3}\right) - \sin^2\left(\arctan\frac{4}{3}\right)$ $\qquad \arctan\frac{4}{3} = \theta$

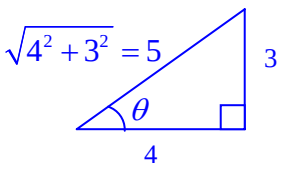
$= \cos^2\theta - \sin^2\theta$ $\qquad \tan\theta = \frac{4}{3}$

$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$

$= \frac{-7}{25}$



31. If $\cot\theta = -\frac{3}{4}$ and θ is in quadrant II, find



a) $\sin\theta = \frac{4}{5}$

b) $\cos\theta = \frac{-3}{5}$

c) $\tan\theta = \frac{-4}{3}$

d) $\sec\theta = \frac{1}{\cos\theta} = \frac{-5}{3}$

e) $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{4}{5}\right)\left(\frac{-3}{5}\right) = \frac{-24}{5}$

f) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{-3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{-7}{25}$

g) $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(\frac{-3}{5}\right)}{2}} = \sqrt{\frac{8}{10}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

$\frac{\pi}{2} < \theta < \pi$

$\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$ $\Rightarrow$ θ is in QI

h) $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\left(\frac{-3}{5}\right)}{2}} = \sqrt{\frac{2}{10}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

i) $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{\frac{-24}{25}}{\frac{-7}{25}} = \dfrac{24}{7}$

j) $\csc\frac{\theta}{2} = \dfrac{1}{\sin\frac{\theta}{2}} = \dfrac{\sqrt{5}}{2}$

k) $\sec 2\theta = \dfrac{1}{\cos 2\theta} = \dfrac{-25}{7}$

l) $\cot\frac{\theta}{2} = \dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \dfrac{\frac{\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}} = \dfrac{1}{2}$

32. If $\sec\theta = \frac{6}{5}$ and $\tan\theta < 0$, find

a) $\sin\theta = \frac{-\sqrt{11}}{6}$ θ is in QIV

b) $\cos\theta = \frac{5}{6}$

c) $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{5}{6}}{\frac{-\sqrt{11}}{6}} = \frac{-5\sqrt{11}}{11}$

d) $\csc\theta = \frac{1}{\sin\theta} = \frac{-6\sqrt{11}}{11}$

e) $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{-\sqrt{11}}{6}\right)\left(\frac{5}{6}\right) = \frac{-5\sqrt{11}}{18}$

f) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{5}{6}\right)^2 - \left(\frac{-\sqrt{11}}{6}\right)^2 = \frac{7}{18}$

g) $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{5}{6}}{2}} = \sqrt{\frac{1}{12}} = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$

$\frac{3\pi}{2} < \theta < 2\pi$

$\frac{3\pi}{4} < \frac{\theta}{2} < \pi$ $\Rightarrow$ θ is in QII

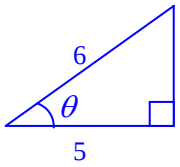
h) $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{5}{6}}{2}} = -\sqrt{\frac{11}{12}} = -\frac{\sqrt{11}}{2\sqrt{3}} = -\frac{\sqrt{33}}{6}$

i) $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{\frac{-5\sqrt{11}}{18}}{\frac{7}{18}} = \dfrac{-5\sqrt{11}}{7}$

j) $\csc\frac{\theta}{2} = \dfrac{1}{\sin\frac{\theta}{2}} = 2\sqrt{3}$

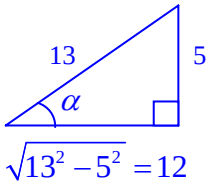
k) $\sec 2\theta = \dfrac{1}{\cos 2\theta} = \dfrac{18}{7}$

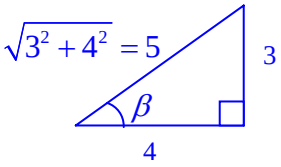
l) $\cot\frac{\theta}{2} = \dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \dfrac{-\frac{\sqrt{33}}{6}}{\frac{\sqrt{3}}{6}} = -\sqrt{11}$



$\sqrt{6^2 - 5^2} = \sqrt{11}$

33. If $\sin\alpha = \frac{-5}{13}$ and $\tan\beta = \frac{-3}{4}$, where α is in quadrant III and β is in quadrant II, find the exact value of the trigonometric function.





a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha = \left(\frac{-5}{13}\right)\left(\frac{-4}{5}\right) + \frac{3}{5}\left(\frac{-12}{13}\right) = \frac{-16}{65}$

b) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha = \left(\frac{-5}{13}\right)\left(\frac{-4}{5}\right) - \frac{3}{5}\left(\frac{-12}{13}\right) = \frac{56}{65}$

c) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(\frac{-12}{13}\right)\left(\frac{-4}{5}\right) - \left(\frac{-5}{13}\right)\left(\frac{3}{5}\right) = \frac{63}{65}$

d) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \left(\frac{-12}{13}\right)\left(\frac{-4}{5}\right) + \left(\frac{-5}{13}\right)\left(\frac{3}{5}\right) = \frac{33}{65}$

e) $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\frac{-16}{65}}{\frac{63}{65}} = \dfrac{-16}{63}$

f) $\csc(\alpha+\beta) = \dfrac{1}{\sin(\alpha+\beta)} = \dfrac{-65}{16}$

g) $\sec(\alpha-\beta) = \dfrac{1}{\cos(\alpha-\beta)} = \dfrac{65}{33}$

h) $\cot(\alpha-\beta) = \dfrac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)} = \dfrac{\frac{33}{65}}{\frac{56}{65}} = \dfrac{33}{56}$

34. Write the first expression in terms of the second if the terminal point determined by θ is in the given quadrant.

a) $\cos\theta$ in terms of $\tan\theta$ if θ in quadrant III.

$$1+\tan^2\theta = \sec^2\theta$$

$$1+\tan^2\theta = \frac{1}{\cos^2\theta}$$

$$\cos^2\theta = \frac{1}{1+\tan^2\theta}$$

$$\cos\theta = \frac{-1}{\sqrt{1+\tan^2\theta}}$$

b) $\sin\theta$ in terms of $\sec\theta$ if θ is in quadrant II.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \frac{1}{\sec^2\theta} = \frac{\sec^2\theta - 1}{\sec^2\theta}$$

$$\sin\theta = \frac{-\sqrt{\sec^2\theta - 1}}{\sec\theta}$$

35. Find the amplitude, period and phase shift and sketch the graph.

a) $f(x)=2\cos 3\left(x-\frac{\pi}{3}\right)$

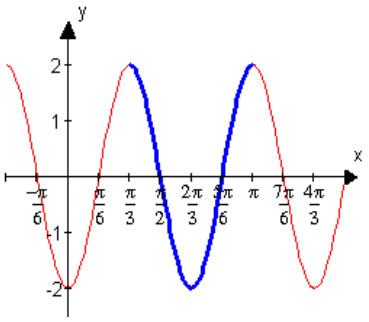
Amplitude: 2

Period: $\frac{2\pi}{3}$

Phase Shift: $\frac{\pi}{3}$

$\frac{\pi}{3}+\frac{2\pi}{3}=\pi$

One period: $\left[\frac{\pi}{3},\pi\right]$



b) $f(x)=-4\sin\left(\frac{x}{3}+\pi\right)=-4\sin\frac{1}{3}(x+3\pi)$

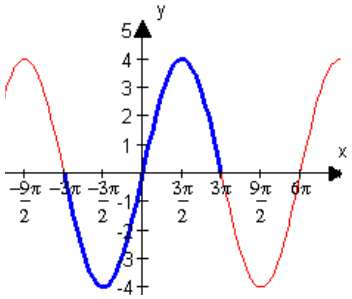
Amplitude: 4

Period: $\frac{2\pi}{\frac{1}{3}}=6\pi$

Phase Shift: -3π

$-3\pi+6\pi=3\pi$

One period: $[-3\pi,3\pi]$



c) $f(x)=2\sin\left(\pi x-\frac{\pi}{4}\right)=2\sin\pi\left(x-\frac{1}{4}\right)$

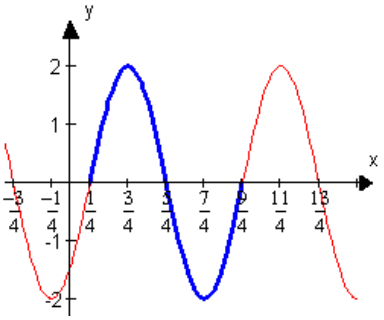
Amplitude: 2

Period: 2

Phase Shift: $\frac{1}{4}$

$\frac{1}{4}+2=\frac{9}{4}$

One period: $\left[\frac{1}{4},\frac{9}{4}\right]$



36. Find the period and phase shift and sketch the graph.

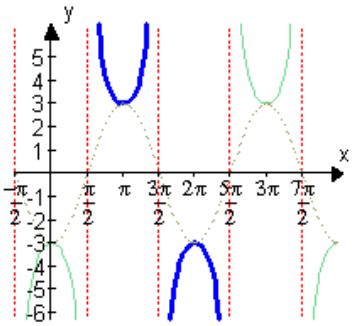
a) $f(x)=3\csc\left(x-\frac{\pi}{2}\right)$

Period: 2π

Phase Shift: $\frac{\pi}{2}$

$\frac{\pi}{2}+2\pi=\frac{5\pi}{2}$

One period: $\left(\frac{\pi}{2},\frac{5\pi}{2}\right)$



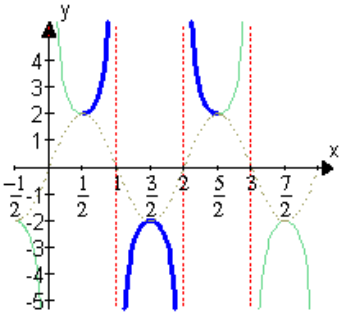
b) $f(x)=2\sec\left(\pi x-\frac{\pi}{2}\right)=2\sec\pi\left(x-\frac{1}{2}\right)$

Period: $\frac{2\pi}{\pi}=2$

Phase Shift: $\frac{1}{2}$

$\frac{1}{2}+2=\frac{5}{2}$

One period: $\left[\frac{1}{2},\frac{5}{2}\right]$



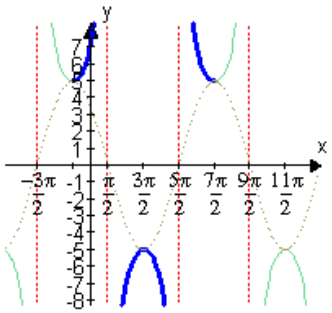
c) $f(x)=5\sec\frac{1}{2}\left(x+\frac{\pi}{2}\right)$

Period: $\frac{2\pi}{\frac{1}{2}}=4\pi$

Phase Shift: $\frac{-\pi}{2}$

$\frac{-\pi}{2}+5\pi=\frac{9\pi}{2}$

One period: $\left[\frac{-\pi}{2},\frac{9\pi}{2}\right]$



d) $f(x)=\tan 2\left(x+\frac{\pi}{3}\right)$

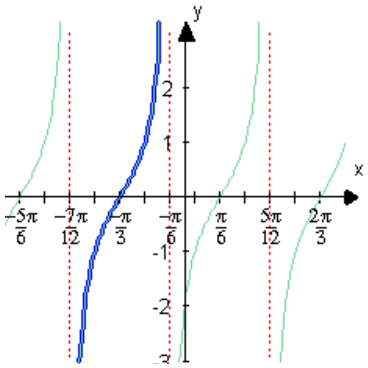
Period: $\frac{\pi}{2}$

Phase Shift: $\frac{-\pi}{3}$

$\frac{-\pi}{3}-\frac{1}{2}\frac{\pi}{2}=\frac{-7\pi}{12}$

$\frac{-\pi}{3}+\frac{1}{2}\frac{\pi}{2}=\frac{-\pi}{6}$

One period: $\left(\frac{-7\pi}{12},\frac{-\pi}{6}\right)$



e) $f(x)=\tan\left(\pi x-\frac{\pi}{3}\right)=\tan\pi\left(x-\frac{1}{3}\right)$

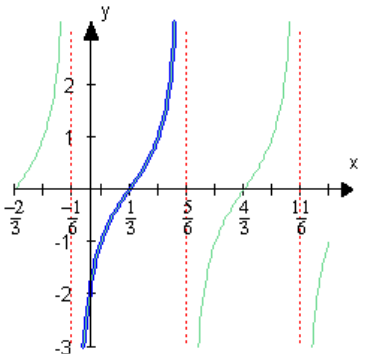
Period: $\frac{\pi}{\pi}=1$

Phase Shift: $\frac{1}{3}$

$\frac{1}{3}-\frac{1}{2}1=\frac{-1}{6}$

$\frac{1}{3}+\frac{1}{2}1=\frac{5}{6}$

One period: $\left(\frac{-1}{6},\frac{5}{6}\right)$



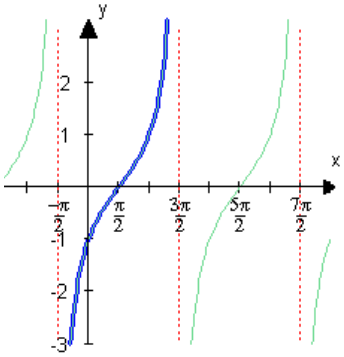
f) $f(x)=-\cot\left(\frac{x}{2}+\frac{\pi}{4}\right)=-\cot\frac{1}{2}\left(x+\frac{\pi}{2}\right)$

Period: $\frac{\pi}{\frac{1}{2}}=2\pi$

Phase Shift: $\frac{-\pi}{2}$

$\frac{-\pi}{2}+2\pi=\frac{3\pi}{2}$

One period: $\left(\frac{-\pi}{2},\frac{3\pi}{2}\right)$



37. Verify each identity.

a) $\dfrac{\csc\theta+\sec\theta}{\sin\theta+\cos\theta}=\cot\theta+\tan\theta$

$$LS=\frac{\csc\theta+\sec\theta}{\sin\theta+\cos\theta}=\frac{\frac{1}{\sin\theta}+\frac{1}{\cos\theta}}{\sin\theta+\cos\theta}=\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}\frac{1}{\sin\theta+\cos\theta}=\frac{1}{\sin\theta\cos\theta}$$

$$RS=\cot\theta+\tan\theta=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}=\frac{1}{\sin\theta\cos\theta}=LS$$

b) $\dfrac{\tan^2\theta+1}{\sec\theta\tan\theta}=\csc\theta$

$$LS=\frac{\tan^2\theta+1}{\sec\theta\tan\theta}=\frac{\sec^2\theta}{\sec\theta\tan\theta}=\frac{\sec\theta}{\tan\theta}=\frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}=\frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}=\frac{1}{\sin\theta}=\csc\theta=RS$$

c) $\dfrac{\sin\theta+1}{\sin\theta}=\dfrac{\cot^2\theta}{\csc\theta-1}$

$$RS=\frac{\cot^2\theta}{\csc\theta-1}=\frac{\csc^2\theta-1}{\csc\theta-1}=\frac{(\csc\theta+1)(\csc\theta-1)}{\csc\theta-1}=\csc\theta+1=\frac{1}{\sin\theta}+1$$

$$=\frac{\sin\theta+1}{\sin\theta}=LS$$

d) $\dfrac{\cos4\theta+\cos2\theta}{\sin4\theta+\sin2\theta}=\cot3\theta$

$$LS=\frac{\cos4\theta+\cos2\theta}{\sin4\theta+\sin2\theta}=\frac{2\cos3\theta\cos\theta}{2\sin3\theta\cos\theta}=\cot3\theta=RS$$

e) $\csc2\theta=\dfrac{\csc\theta}{2\cos\theta}$

$$LS=\csc2\theta=\frac{1}{\sin2\theta}=\frac{1}{2\sin\theta\cos\theta}=\frac{\csc\theta}{2\cos\theta}=RS$$

f) $\cos^4\theta-\sin^4\theta=\cos2\theta$

$$LS=\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)=\cos2\theta=RS$$

g) $\sec^2\frac{\theta}{2}=\dfrac{2(1-\cos\theta)}{\sin^2\theta}$

$$LS=\sec^2\frac{\theta}{2}=\frac{1}{\cos^2\frac{\theta}{2}}=\frac{1}{\frac{1+\cos\theta}{2}}=\frac{2}{1+\cos\theta}$$

$$RS=\frac{2(1-\cos\theta)}{\sin^2\theta}=\frac{2(1-\cos\theta)}{1-\cos^2\theta}=\frac{2(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}=\frac{2}{1+\cos\theta}=LS$$

h) $\tan\theta=\dfrac{1-\cos2\theta}{\sin2\theta}$

$$RS=\frac{1-\cos2\theta}{\sin2\theta}=\frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}=\frac{2\sin^2\theta}{2\sin\theta\cos\theta}=\frac{\sin\theta}{\cos\theta}=\tan\theta=LS$$

i) $\sec(A-B)=\dfrac{\csc A\csc B}{1+\cot A\cot B}$

$$LS=\sec(A-B)=\frac{1}{\cos(A-B)}=\frac{1}{\cos A\cos B+\sin A\sin B}\cdot\frac{\frac{1}{\sin A\sin B}}{\frac{1}{\sin A\sin B}}$$

$$=\frac{\frac{1}{\sin A\sin B}}{\frac{\cos A\cos B}{\sin A\sin B}+\frac{\sin A\sin B}{\sin A\sin B}}=\frac{\csc A\csc B}{\cot A\cot B+1}=RS$$

j) $\cot\left(\dfrac{A+B}{2}\right)=\dfrac{\sin A-\sin B}{\cos B-\cos A}$

$$RS=\frac{\sin A-\sin B}{\cos B-\cos A}=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}=\cot\tfrac{A+B}{2}=LS$$

38. a) Write $\cos 2x\sin 5x$ as a sum of trigonometric functions.

$$\cos 2x\sin 5x=\tfrac{1}{2}\left(\sin(7x)-\sin(-3x)\right)=\tfrac{1}{2}\sin 7x+\tfrac{1}{2}\sin 3x$$

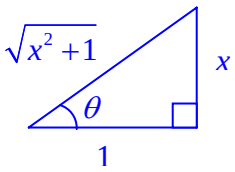
b) Write $\cos 2x+\cos 5x$ as a product of trigonometric functions.

$$\cos 2x+\cos 5x=2\cos\tfrac{7x}{2}\cos\tfrac{-3x}{2}=2\cos\tfrac{7x}{2}\cos\tfrac{3x}{2}$$

39. Rewrite the expression as an algebraic function of x.

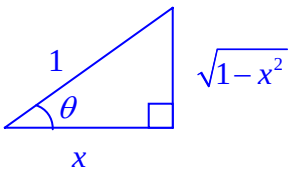
a) $\sin(\arctan x)=\sin\theta$ $\qquad \arctan x=\theta$

$$=\frac{x}{\sqrt{1+x^2}} \qquad \tan\theta=x$$



b) $\cot(\arccos x)=\cot\theta$ $\qquad \arccos x=\theta$

$$=\frac{x}{\sqrt{1-x^2}} \qquad \cos\theta=x$$



40. Solve each equation on the interval $[0,2\pi)$.

a) $\tan\theta=-\frac{\sqrt{3}}{3}$

$\theta=\frac{5\pi}{6},\frac{11\pi}{6}$

b) $\sin 3\theta=-1$

$3\theta=\frac{3\pi}{2},\frac{7\pi}{2},\frac{11\pi}{2},\ldots$

$\theta=\frac{\pi}{2},\frac{7\pi}{6},\frac{11\pi}{6}$

c) $\sec\frac{3\theta}{2}=-2$

$\cos\frac{3\theta}{2}=\frac{-1}{2}$

$\frac{3\theta}{2}=\frac{2\pi}{3},\frac{4\pi}{3},\frac{8\pi}{3},\ldots$

$\theta=\frac{4\pi}{9},\frac{8\pi}{9},\frac{16\pi}{9}$

d) $2\cos^2\theta-\cos\theta-1=0$

$(2\cos\theta+1)(\cos\theta-1)=0$

$2\cos\theta+1=0$ $\qquad \cos\theta-1=0$

$\cos\theta=-\frac{1}{2}$ $\qquad \cos\theta=1$

$\theta=\frac{2\pi}{3},\frac{4\pi}{3}$ $\qquad \theta=0$

Ergo, $\theta=0,\frac{2\pi}{3},\frac{4\pi}{3}$

e) $\tan\theta = 2\sin\theta$

$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$$

$$\sin\theta = 2\sin\theta\cos\theta$$

$$\sin\theta - 2\sin\theta\cos\theta = 0$$

$$\sin\theta(1-2\cos\theta) = 0$$

$\sin\theta = 0$ $\qquad$ $1-2\cos\theta = 0$

$\theta = 0, \pi$ $\qquad$ $\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Ergo, $\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

f) $\sin 2\theta + \sin 4\theta = 0$

$$2\sin\tfrac{2\theta+4\theta}{2}\cos\tfrac{2\theta-4\theta}{2} = 0$$

$$2\sin 3\theta\cos(-\theta) = 0$$

$$2\sin 3\theta\cos\theta = 0$$

$\sin 3\theta = 0$ $\qquad$ $\cos\theta = 0$

$3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots$ $\qquad$ $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

Ergo, $\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

g) $\tan^2\theta = \frac{3}{2}\sec\theta$

$$\sec^2\theta - 1 = \tfrac{3}{2}\sec\theta$$

$$2\sec^2\theta - 3\sec\theta - 2 = 0$$

$$(2\sec\theta + 1)(\sec\theta - 2) = 0$$

$2\sec\theta + 1 = 0$ $\qquad$ $\sec\theta - 2 = 0$

$\sec\theta = -\frac{1}{2}$ $\qquad$ $\sec\theta = 2$

$\cos\theta = -2$ $\qquad$ $\cos\theta = \frac{1}{2}$

No solution $\qquad$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Ergo, $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

41. Solve the following triangles.

a) $A = 20°$, $a = 5$, $b = 8$.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b\sin A}{a} = \frac{8\sin 20°}{5}$$

$B \simeq 33.2°$ $\qquad$ $B_2 = 180° - B \simeq 146.8°$

$C = 180° - A - B$ $\qquad$ $C_2 = 180° - A - B_2$

$= 180° - 20° - 33.2°$ $\qquad$ $= 180° - 20° - 146.8°$

$= 126.8°$ $\qquad$ $= 13.2°$

$\frac{\sin C}{c} = \frac{\sin A}{a}$ $\qquad$ $\frac{\sin C_2}{c_2} = \frac{\sin A}{a}$

$c = \frac{a\sin C}{\sin A} = \frac{5\sin 126.8°}{\sin 20°}$ $\qquad$ $c_2 = \frac{a\sin C_2}{\sin A} = \frac{5\sin 13.2°}{\sin 20°}$

$\simeq 11.7$ $\qquad$ $\simeq 3.23$

b) $A = 24.3°,\ C = 54.6°,\ c = 2.6$

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$a = \frac{c\sin A}{\sin C} = \frac{2.6\sin 24.3°}{\sin 54.6°} \simeq 1.31$$

$$B = 180° - A - C$$

$$= 180° - 24.3° - 54.6°$$

$$= 101.1°$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$b = \frac{c\sin B}{\sin C} = \frac{2.6\sin 101.1°}{\sin 54.6°} \simeq 3.3$$

c) $a = 55,\ b = 52,\ c = 72$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{52^2 + 72^2 - 55^2}{2(52)(72)} = \frac{1621}{2496}$$

$$A \simeq 49.5°$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{55^2 + 72^2 - 52^2}{2(55)(72)} = \frac{367}{528}$$

$$B \simeq 46.0°$$

$$C = 180° - A - B$$

$$= 180° - 49.5° - 46°$$

$$= 84.5°$$

d) $A = 55°,\ b = 3,\ c = 10$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= 3^2 + 10^2 - 2(3)(10)\cos 55°$$

$$\simeq 74.5854$$

$$a \simeq 8.64$$

$$C = 180° - A - B$$

$$= 180° - 55° - 16.5°$$

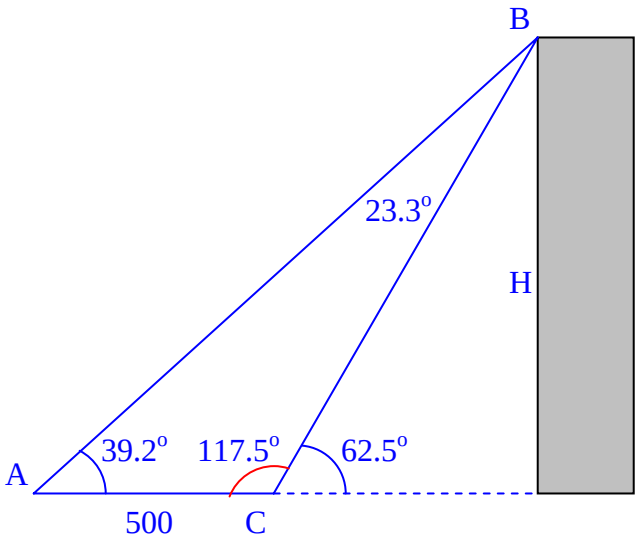
$$= 108.5°$$

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b\sin A}{a} = \frac{3\sin 55°}{8.64}$$

$$B \simeq 16.5°$$

42. From a point *A* on the ground, the angle of elevation to the top of a skyscraper is 39.2^o. From a point *B*, 500m from point *A*, the angle of elevation is 62.5^o. Find the height of the skyscraper.



$$C = 180° - 62.5° = 117.5°$$
$$B = 180° - A - C = 180° - 39.2° - 117.5° = 23.3°$$
$$\frac{\sin C}{c} = \frac{\sin B}{b}$$
$$c = \frac{b \sin C}{\sin B} = \frac{500 \sin 117.5°}{\sin 23.3°} \simeq 1121.25$$
$$\sin A = \frac{H}{c}$$
$$H = c \sin A = 1121.25 \sin 39.2 = 708.7\text{m}$$